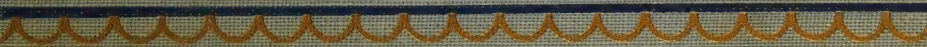




Analytic Geometry



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ANALYTIC GEOMETRY



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ANALYTIC GEOMETRY

BY

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Third Edition

THE MACMILLAN COMPANY

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WRITTEN FOR INCLUSION IN MAGAZINE OR NEWSPAPER

— PRINTED IN THE UNITED STATES OF AMERICA —

Set up and electrotyped. Published June, 1938.
Third Printing, December, 1939.

Over 57,000 copies printed since publication

First and second editions copyrighted and published, 1923 and 1927,
By The Macmillan Company.

PREFACE

The third edition differs from its predecessor in a number of important respects. A bird's-eye view may be presented as follows.

Arts. 6, 11, 30–31 are new; certain parts of Chap. IV have been somewhat condensed. Otherwise the first four chapters are substantially unchanged.

The amount of space devoted to the conic sections, including the circle, has been reduced by about one-third. This has been accomplished partly by omission, partly by more effective arrangement. For instance, the problem of tangents by the quadratic discriminant appears in only one place—Arts. 80–85. Arts. 51, 70 are new. A variety of definitions of the conics are introduced early—Arts. 44–46, 53, 59.

In the chapter on coördinate transformations, Arts. 64–68 are new.

Chaps. X, XIII, XIV, on algebraic curves, “special curves,” and curve-fitting respectively, are entirely new. The treatment of polar coördinates (Chap. XI) and parametric equations (Chap. XII) has been materially strengthened.

There is a slight reduction in the amount of space devoted to solid geometry, some saving being effected by reorganization, some by abbreviation—notably in the treatment of the standard quadrics. Arts. 186–187, 189–190 are new.

The writer has kept in mind at all times the instructor who, for lack of time, must omit many topics. The ar-

rangement is such that the less vital material may be omitted without loss of continuity.

The drill exercises have been almost entirely worked over. The number of exercises is much greater, of course, than any one class can be expected to solve. They are arranged so that the odd and even numbers form two sets of nearly equal value, which makes it a simple matter to assign homework in two successive years with very little repetition. As a help to the student, the number of text-references and other hints has been considerably increased.

CLYDE E. LOVE

ANN ARBOR, MICHIGAN
May, 1938

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ANALYTIC GEOMETRY

PLANE ANALYTIC GEOMETRY

CHAPTER I

CARTESIAN COÖRDINATES

1. Directed line segments. When a line segment is measured in a definite sense *from* one endpoint *to* the other, the segment is said to be *directed*. If the terminal points are A, B , we speak of the segment AB or the segment BA according as the sense is from A to B or from B to A .

If one sense is chosen as positive, then the opposite sense is negative: thus

$$AB = -BA, \quad \text{or} \quad AB + BA = 0.$$

If C is any third point of the straight line through A and B , then for all possible positions of A, B , and C we have

$$AB + BC = AC,$$

or

$$AB + BC + CA = 0.$$



FIG. 1

Two directed line segments lying in the same line or in parallel lines are said to be *equal* if they have the same length and are measured in the same sense.

2. Position of a point on a surface. If a point lies on a given surface, two magnitudes, or “coördinates,” are necessary to determine its position, each coördinate being measured in a definite sense. Thus the position of a picture on a wall may be given by its distance to the right (or left) of a window and its height above the floor.

3. Cartesian coördinates. Given a point P lying in a certain plane, let us assume two perpendicular lines Ox , Oy lying in the plane. The line Ox is called the x -axis, Oy the y -axis, and their point of intersection O is the *origin*. The

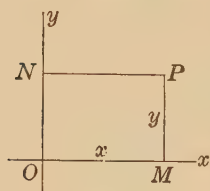


FIG. 2

position of P is evidently known if its distances from the axes are given, each being measured in a definite sense. These directed segments are called the *cartesian coördinates* of P : the distance from the y -axis— NP or its equal OM —is the *abscissa*, the distance MP from the x -axis is the *ordinate*. Each coördinate is represented algebraically by a *number*.

We shall ordinarily assume the axes as in Fig. 2, and shall consider abscissas *positive* if measured *to the right*, *negative to the left*; ordinates *positive* if measured *upward*, *negative downward*.

It is customary to write the coördinates of a point in parentheses, with the abscissa first: thus in Fig. 3 the point $P : (3, 5)$, also written simply $(3, 5)$, has the abscissa 3 and the ordinate 5. The figure also shows the points $Q : (2, -4)$, $R : (-4, -3)$, and $S : (-3, 0)$.

The axes divide the plane into four compartments, called *quadrants*, and numbered as in Fig. 3. Thus the abscissa is positive in the first and fourth quadrants, the ordinate positive in the first and second.

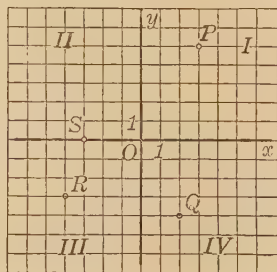


FIG. 3

In all work with cartesian coördinates, it will in general be assumed that segments oblique to the axes are *undirected*, segments parallel to an axis are *directed*. Segments

parallel to Ox will be considered positive if measured to the right, negative to the left; segments parallel to Oy , positive upward, negative downward.

By the introduction of a cartesian coördinate system there is set up a unique correspondence between *points*, on the one hand, and *pairs of real numbers*, on the other: *to every pair of real numbers there corresponds one and only one point in the plane, and conversely*.*

4. Units. In analytic geometry, drawings are usually made on square-ruled paper, called *coördinate paper* (see Fig. 3). The unit of measurement chosen need not, and usually should not, be the width of one space on the coördinate paper; the scale should be selected with regard to the nature of the drawing to be made—neither so large that some of the points fall beyond the limits of the paper, nor so small that the properties of the figure become obscured. The scale adopted should be clearly indicated.

Cases sometimes arise in which it is convenient to adopt different scales on the two axes, which of course produces a distortion of the figure (see Exs. 37–40 below). Except where the contrary is stated we shall assume always that the unit for ordinates is the same as that for abscissas.

To plot a point whose coördinates are irrational, we employ decimal approximations. For instance, to plot the point $(\sqrt{2}, \sqrt{3})$, we might take †

$$\sqrt{2} = 1.41, \sqrt{3} = 1.73.$$

* The coördinates defined above are more precisely called *rectangular cartesian coördinates*. It is possible to set up a system in which the axes are oblique to each other, but in this book only the rectangular system will be used.

The word *cartesian* is derived from the name of René Descartes (1596–1650), who was the founder of analytic geometry.

† Of course $(\sqrt{2}, \sqrt{3})$ and $(1.41, 1.73)$ are not at all the same point—we are merely doing the best we can for plotting purposes. Such approximations are permissible only in plotting, which is an approximate process at best, or in applications where an approximate result is satisfactory.

5. Distance between two points. The distance between two points P_1, P_2 can be expressed in terms of their coördinates by the Theorem of Pythagoras. Let the coördinates of the two points be denoted by the letters x, y with subscripts: $P_1 : (x_1, y_1), P_2 : (x_2, y_2)$. Now, in Fig. 4,

$$(1) \quad d = \sqrt{P_1Q^2 + QP_2^2}.$$

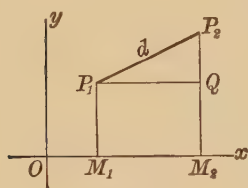


FIG. 4

But

$$P_1Q = M_1M_2 = OM_2 - OM_1, \\ QP_2 = M_2P_2 - M_2Q:$$

that is,

$$P_1Q = x_2 - x_1, \\ QP_2 = y_2 - y_1.$$

Substituting in (1), we find

$$(2) \quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

By drawing the figure in various positions, the student may convince himself that the formula holds no matter where the points P_1, P_2 may be situated—not merely when both points lie in the first quadrant.

Examples: (a) Find the distance between the points $(3, 2)$ and $(-5, 4)$.

By formula (2), or directly from a figure, we find

$$d = \sqrt{(-8)^2 + 2^2} = \sqrt{68} = 2\sqrt{17}.$$

(b) A point moves so that its distance from the origin is always 2. Express this fact by an algebraic equation.

Let the coördinates of the moving point be (x, y) . Then, by (2), the distance of this point from the origin is $\sqrt{x^2 + y^2}$. Hence the required equation is

$$\sqrt{x^2 + y^2} = 2, \quad \text{or} \quad x^2 + y^2 = 4.$$

The locus of the point (x, y) is evidently a circle of radius 2 with center at O .

EXERCISES

Draw the following figures on coördinate paper, choosing a suitable scale in each instance.

1. Triangle with vertices $(1, 3)$, $(5, 0)$, $(-2, -1)$.
2. Quadrilateral with vertices $(5, -2)$, $(4, 3)$, $(-2, 3)$, $(2, -5)$.
3. Quadrilateral with vertices $(\frac{1}{2}, -\frac{7}{8})$, $(\frac{5}{8}, 0)$, $(-\frac{1}{4}, 0)$, $(-\frac{1}{2}, -\frac{1}{8})$.
4. Triangle with vertices $(24, 12)$, $(-30, 6)$, $(15, -15)$.
5. What can be said of the coördinates of all points on the x -axis? On the y -axis? On the line through O bisecting the first and third quadrants? The second and fourth quadrants? On the line parallel to the y -axis two units to the right of it? Two units to the left?

6. Where does a point lie if its abscissa is 0? If its ordinate is 0? If abscissa and ordinate are equal? Are numerically equal but of opposite sign?

Find the distance between the given points.

7. $(5, 3)$, $(6, 7)$. *Ans.* $\sqrt{17}$.
8. $(-6, 2)$, $(-4, -3)$. *Ans.* $\sqrt{29}$.
9. $(\frac{3}{2}, -\frac{7}{4})$, $(-\frac{3}{4}, -\frac{5}{2})$. *Ans.* $\frac{3}{4}\sqrt{10}$.
10. $(\frac{5}{8}, -\frac{1}{2})$, $(\frac{1}{4}, -\frac{4}{3})$. *Ans.* $\frac{1}{12}\sqrt{149}$.
11. Prove that the points $(5, 0)$, $(2, 1)$, $(4, 7)$ are the vertices of a right triangle, and find its area. *Ans.* $A = 10$.
12. Prove that the points $(-2, 1)$, $(0, -5)$, $(10, 5)$ are the vertices of a right triangle, and find its area. *Ans.* $A = 40$.
13. Prove that the points $(3, 2)$, $(7, -2)$, $(6, 1)$ are the vertices of an isosceles triangle, and find its area. *Ans.* $A = 4$.
14. Prove that the points $(1, 3)$, $(3, -1)$, $(7, -3)$ are the vertices of an isosceles triangle, and find its area. *Ans.* $A = 6$.
15. Prove that the points $(-7, 1)$, $(5, -4)$, $(10, 8)$, $(-2, 13)$ are the vertices of a square, and find its area. *Ans.* $A = 169$.
16. Prove that the points $(-\frac{3}{2}, 4)$, $(-\frac{7}{2}, 3)$, $(-\frac{3}{2}, -1)$, $(\frac{1}{2}, 0)$ are the vertices of a parallelogram. Is the parallelogram a rectangle?
17. Draw the circle with center at $(3, 2)$ passing through $(13, -10)$. Does this circle pass through $(-11, 9)$?
18. Draw a circle with center at $(0, -13)$ tangent to the x -axis. Does this circle pass through $(11, -6)$? Through $(-5, -1)$?
19. Find the radius of a circle with center at $(3, 1)$, if a chord of length 6 is bisected at $(6, 5)$. *Ans.* $\sqrt{34}$.
20. Find the radius of a circle with center at $(1, -1)$, if a chord of length 10 is bisected at $(2, 0)$.

21. The center of a circle is at (5, 3) and its radius is 5. Find the length of the chord that is bisected at (3, 2). *Ans.* $4\sqrt{5}$.

22. The center of a circle is at (6, -1) and its radius is 6. Find the length of the chord that is bisected at (3, 4). *Ans.* $2\sqrt{2}$.

23. Prove that the quadrilateral with vertices (0, 4), (7, -7), (2, -2), (1, -9) consists of two equal triangles placed base to base, and find its area. *Ans.* $A = 20$.

24. Prove that the points (6, 2), (13, 1), (12, -6), (1, -8) form a kite-shaped quadrilateral, and find its area. *Ans.* $A = 75$.

By division into triangles in a suitable way, find the area of the quadrilateral having the given points as vertices.

25. (8, 4), (7, -3), (-4, -5), (1, 5). *Ans.* 75.

26. (3, 2), (-8, 9), (-3, 4), (-10, 3). *Ans.* 20.

27. (0, 0), (-3, 4), (-7, 1), (-6, -6).

28. (-1, 3), (0, 5), (-7, -4), (9, -2).

Determine whether the given points lie in a straight line.

29. (-2, -8), (1, -7), (10, -4). 30. (3, 8), (5, 4), (6, 2).

31. (-5, -5), (5, 2), (12, 7). 32. (-2, 12), (9, 6), (22, -1).

Express the given statement by means of an algebraic equation. What is the locus of the point (x, y) in each case?

33. The point (x, y) is at the distance 3 from (6, -2).

34. The point (x, y) is at the distance 2 from (-5, 7).

35. The point (x, y) is equidistant from (4, 2) and (-3, 3).

Ans. $7x - y = 1$.

36. The point (x, y) is equidistant from (-3, 5) and (-4, -2).

Draw the following figures, with the y -scale twice as large as the x -scale.

37. The square with vertices (0, 0), (1, 0), (1, 1), (0, 1).

38. The right triangle of Ex. 12.

39. The isosceles triangle of Ex. 13.

40. The square of Ex. 15.

6. Division of a line-segment. Given two points $P_1 : (x_1, y_1)$, $P_2 : (x_2, y_2)$, let it be required to find the point $P : (x, y)$ lying in the line joining P_1, P_2 and so placed that the segment P_1P is a given fraction of the entire segment P_1P_2 : say

$$P_1P = k \cdot P_1P_2.$$

By similar triangles,

$$\frac{M_1M}{M_1M_2} = \frac{P_1P}{P_1P_2} = k,$$

or

$$M_1M = k \cdot M_1M_2.$$

But

$$M_1M_2 = x_2 - x_1,$$

so that

$$x = OM_1 + M_1M = x_1 + k(x_2 - x_1).$$

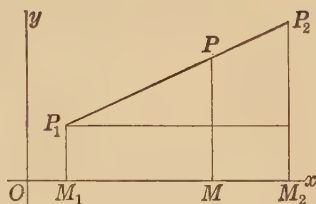


FIG. 5

A similar formula for y is readily derived.

If P lies in the segment P_1P_2 produced, rather than in the interior of the segment, then obviously $k > 1$, but the result is the same. Hence:

If $P : (x, y)$ is a point in the line-segment P_1P_2 or in P_1P_2 produced, and if P is so placed that $P_1P = k \cdot P_1P_2$, then

$$(1) \quad x = x_1 + k(x_2 - x_1), \quad y = y_1 + k(y_2 - y_1).$$

Examples: (a) The segment from $(1, 3)$ to $(5, -2)$ is trisected. Find the point of trisection nearer to $(1, 3)$.

Here $k = \frac{1}{3}$, $x_2 - x_1 = 4$, $y_2 - y_1 = -5$, so that

$$x = 1 + \frac{1}{3} \cdot 4 = \frac{7}{3}, \quad y = 3 + \frac{1}{3} \cdot (-5) = \frac{4}{3}.$$

Thus the required point is $(\frac{7}{3}, \frac{4}{3})$.

(b) A line is drawn from $(5, -4)$ to $(7, -9)$, and is then extended beyond the latter point so that its length is doubled. Find the terminal point.

Here $k = 2$, $x_2 - x_1 = 2$, $y_2 - y_1 = -5$, so that

$$x = 5 + 4 = 9, \quad y = -4 - 10 = -14.$$

7. Midpoint of a line-segment. An important special case of the problem of § 6 is that in which P is the *midpoint* of the segment P_1P_2 . Putting $k = \frac{1}{2}$ in (1) above, we find by very simple algebra that:

The coördinates (x, y) of the point midway between (x_1, y_1) , (x_2, y_2) are

$$(1) \quad x = \frac{1}{2}(x_1 + x_2), \quad y = \frac{1}{2}(y_1 + y_2).$$

EXERCISES

1. Trisect the segment joining $(7, -2)$, $(3, 5)$. *Ans.* $(\frac{17}{3}, \frac{1}{3})$; $(\frac{13}{3}, \frac{8}{3})$.
2. Trisect the segment joining $(-1, -3)$, $(6, 4)$.
3. The segment joining $(6, 14)$, $(2, -2)$ is to be divided into four equal parts. Find the points of division.
4. The segment joining $(5, 0)$, $(4, 3)$ is divided into two segments, one of which is three-fourths of the other. Find the point of division.
Ans. $(\frac{32}{7}, \frac{9}{7})$; $(\frac{31}{7}, \frac{12}{7})$.
5. The segment joining $(3, -2)$, $(10, 12)$ is divided into two segments, one of which is two-fifths of the other. Find the point of division.
6. The segment from $(-1, 4)$ to $(6, 2)$ is trebled. Find the terminal point.
7. The segment joining $(4, 3)$, $(6, -1)$ is extended each way a distance equal to one-half its own length. Find the terminal points.
8. The center of a circle is $(4, 7)$; one point is $(5, -3)$. Find the other end of the diameter through this point.

Using (1), § 7, find the point midway between the given points.

9. $(3, 5)$, $(-1, 6)$.
10. $(-4, -3)$, $(-7, -2)$.
11. $(8, -7)$, $(-5, 3)$.
12. $(-\frac{5}{2}, \frac{3}{4})$, $(\frac{3}{2}, \frac{5}{2})$.
13. Prove in two ways that the points $(10, 4)$, $(3, -5)$, $(1, 1)$ form a right triangle. Find the area of the circumscribed circle. *Ans.* $\frac{65}{2}\pi$.
14. Prove in two ways that the points $(2, 4)$, $(-2, 3)$, $(8, -20)$ form a right triangle. Find the area of the circumscribed circle. *Ans.* $\frac{629}{4}\pi$.
15. Prove in two ways that the points $(\frac{3}{2}, 1)$, $(2, 5)$, $(3, -2)$, $(\frac{5}{2}, -6)$ form a parallelogram.
16. Prove in two ways that the points $(1, 2)$, $(-5, 1)$, $(-6, -\frac{1}{2})$, $(0, \frac{1}{2})$ form a parallelogram.
17. Three consecutive vertices of a parallelogram are $(4, 2)$, $(5, 3)$, $(6, -4)$. Find the fourth vertex.

8. Inclination; slope. The *angle of inclination*, also called simply the *inclination*, of a straight line is the (positive or negative) *acute angle* α *between the line and the positive x -axis*, this angle being measured *from Ox to the line*.

The *slope* of a line is the *tangent of the angle of inclination*. Slope is usually denoted by m :

$$(1) \quad m = \tan \alpha.$$

If the axes are in the conventional position, a line sloping *upward to the right* has *positive slope*, since the tangent of a positive acute angle is positive; a line sloping *downward to the right* has *negative slope*.

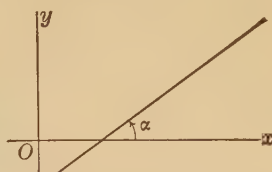


FIG. 6

Given two points $P_1 : (x_1, y_1)$, $P_2 : (x_2, y_2)$ on the line (Fig. 7), we have

$$m = \tan \alpha = \frac{QP_2}{P_1Q}.$$

It follows (§ 5) that:

The slope of the line joining the points $P_1 : (x_1, y_1)$ and $P_2 : (x_2, y_2)$ is

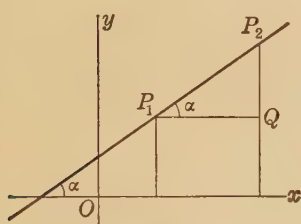


FIG. 7

$$(2) \quad m = \frac{y_2 - y_1}{x_2 - x_1}.$$

In ordinary language, the “slope” of a line means the ratio of “rise” to “run”—i.e. the *ratio of the vertical distance to the horizontal distance* covered in traversing any segment of the line. Thus a road with 10% “slope,” or “grade,” rises 10 ft. for every

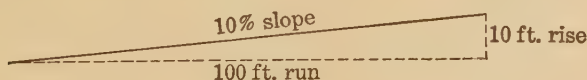


FIG. 8

100 ft. horizontally (Fig. 8). Evidently this is exactly equivalent to the definition given above.

When the scales in the two directions are not the same, the defining formula (1) is interpreted as *equivalent to* (2).

It should be noted that the idea of slope is meaningless in the case of a line parallel to the y -axis (including the y -axis itself), since $\tan 90^\circ$ is non-existent. For this reason, it will be understood throughout this book that in all discussions involving slopes, lines parallel to Oy are excluded.

9. Parallel and perpendicular lines. If two lines are parallel, they evidently have the same slope; and conversely.

Given two perpendicular lines λ_1, λ_2 , with slopes

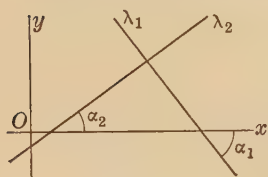


FIG. 9

$$m_1 = \tan \alpha_1, \quad m_2 = \tan \alpha_2,$$

we note that $\alpha_2 = \alpha_1 + 90^\circ$, so that, by trigonometry,

$$\tan \alpha_2 = -\cot \alpha_1 = -\frac{1}{\tan \alpha_1},$$

or

$$(1) \quad m_2 = -\frac{1}{m_1}.$$

Thus we have the

THEOREM: *If two lines are perpendicular, the slope of one is the negative reciprocal of the slope of the other; or in other words, the product of their slopes is -1 .*

Here also the converse is true, as may easily be shown.

Example: Prove that the points $P_1 : (-1, 3)$, $P_2 : (0, 5)$, $P_3 : (3, 1)$ form a right triangle.

From the figure we see that if there is a right angle it must be at P_1 . The slopes of P_1P_2 , P_1P_3 are respectively

$$m_1 = \frac{5 - 3}{0 + 1} = 2, \quad m_2 = \frac{1 - 3}{3 + 1} = -\frac{1}{2},$$

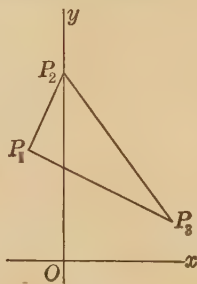


FIG. 10

whence it follows that P_1P_2 and P_1P_3 are perpendicular.

EXERCISES

Find the slope of the line joining the given points. Draw the figure.

1. $(3, -4), (-1, 2)$.
2. $(5, 3), (3, -2)$.
3. $(-6, 2), (2, -5)$.
4. $(-7, -5), (-3, 0)$.
5. $(\frac{3}{2}, \frac{5}{2}), (-\frac{1}{2}, -\frac{1}{2})$.
6. $(-\frac{5}{4}, \frac{3}{4}), (\frac{1}{4}, \frac{3}{8})$.
7. Prove that the slope of any line parallel to the x -axis is 0.
8. State and prove the converse of the theorem of Ex. 7.

Test each of the following statements (Exs. 9–16) by a method based on §§ 8–9.

9. The points $(-1, 1), (5, -1), (3, 13)$ form a right triangle.
10. The points $(1, 3), (6, 5), (5, -7)$ form a right triangle.
11. The points $(2, 4), (3, 8), (5, 1), (4, -3)$ form a parallelogram.
12. The points $(-5, 3), (7, -12), (12, 24), (0, 39)$ form a parallelogram.
13. The circle having the points $(7, 2), (-3, 2)$ as ends of a diameter also passes through $(-1, 6)$.
14. The perpendicular bisector of the line joining $(-3, 1), (13, 3)$ passes through $(7, -14)$.
15. The points $(1, -9), (7, 6), (3, -4)$ lie in a straight line.
16. The points $(0, 2), (4, 1), (16, -2)$ lie in a straight line.
17. Solve Ex. 31, p. 6, by a new method.
18. Solve Ex. 32, p. 6, by a new method.
19. Prove that the quadrilateral with vertices $(0, 1), (4, 2), (3, 6), (-5, 4)$ has two right angles. Find the area. *Ans.* $A = \frac{51}{2}$.
20. Prove that the quadrilateral with vertices $(10, 10), (-14, -2), (-10, -10), (4, -24)$ can be divided into two right triangles. Find the area.

21. Prove that the points $(17, 6), (33, -6), (32, -24), (0, 0)$ form an isosceles trapezoid. Find the area. *Ans.* $A = 450$.

Draw the following lines, (a) with equal scales in the two directions; (b) with the x -scale twice the y -scale; (c) with the y -scale three times the x -scale. (See the next-to-last paragraph of § 8.)

22. Through $(0, 0)$ with slope 1.
23. Through $(0, 0)$ with slope -2 .
24. Through $(3, 1)$ with slope $-\frac{2}{3}$.
25. Through $(-4, 2)$ with slope $\frac{1}{3}$.
26. State and prove the converse of the theorem of § 9.

10. Area of a triangle. In Fig. 11, if from the area of the trapezoid $M_3M_2P_2P_3$ we subtract the areas of the trapezoids $M_3M_1P_1P_3$ and $M_1M_2P_2P_1$, there remains the area of the triangle $P_1P_2P_3$. Let the points P_1, P_2, P_3 be $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. Then

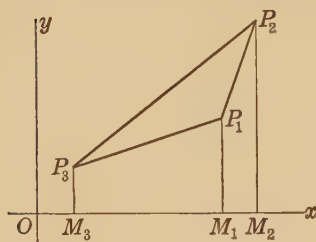


FIG. 11

$$\begin{aligned} M_3M_2P_2P_3 &= \frac{1}{2}(x_2 - x_3)(y_2 + y_3), \\ M_3M_1P_1P_3 &= \frac{1}{2}(x_1 - x_3)(y_1 + y_3), \\ M_1M_2P_2P_1 &= \frac{1}{2}(x_2 - x_1)(y_2 + y_1). \end{aligned}$$

Thus

$$A = \frac{1}{2}[(x_2 - x_3)(y_2 + y_3) - (x_1 - x_3)(y_1 + y_3) - (x_2 - x_1)(y_2 + y_1)],$$

or after rearranging,

$$(1) \quad A = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)].$$

We see by inspection that this may be written in the form

$$(2) \quad A = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

since the right member of (1) is merely the expansion, by minors of the first column, of the determinant (2).

The area of a given triangle as found by this formula may be either positive or negative, according to the order in which the points are taken. We shall in general consider area as essentially positive, using the formula to find the number of square units regardless of sign.

EXERCISES

Find the area of the triangle.

1. $(3, 2), (-5, 3), (6, -2)$.

2. $(-4, 1), (-5, 6), (8, 2)$.

3. $(0, -3), (4, 2), (5, -4)$.

4. $(4, 0), (2, 3), (-8, -6)$.

5. (4, 0), (6, -1), (10, -3). Interpret the result.

6. (3, 5), (-1, 7), (-3, 8). Interpret the result.

Find the area of the quadrilateral having the given points as consecutive vertices. In each case, check by dividing into triangles in two ways.

7. (-4, 0), (-2, 4), (7, 4), (1, -6).

8. (6, 1), (5, 2), (3, 3), (-7, -2).

9. (-6, 1), (9, -5), (6, 3), (5, -2).

10. Solve Ex. 29, p. 6, in a new way.

11. Solve Ex. 30, p. 6, in a new way.

12. Three of the vertices of a rectangle are (3, -2), (7, -5), (5, -6).

Find the area by two methods.

13. Find the fourth vertex of the rectangle of Ex. 12.

14. Three vertices of a parallelogram are (2, 1), (3, 2), (4, -5). Find the area.

15. Find the fourth vertex of the parallelogram of Ex. 14.

16. Find the distance of the point (-1, 6) from the line drawn through the points (1, 2), (6, -3). *Ans.* $\sqrt{2}$.

17. Find the distance of the point (4, 6) from the line drawn through the points (2, 9), (-2, -3). *Ans.* $\frac{9}{10}\sqrt{10}$.

18. Find the distance of the point (5, 2) from the line through (-5, -5), (12, 7). (Ex. 31, p. 6.) *Ans.* $\frac{1}{\sqrt{433}}$.

19. Find the distance of the point (9, 6) from the line through (-2, 12), (22, -1). (Ex. 32, p. 6.) *Ans.* $\frac{1}{\sqrt{745}}$.

20. Prove that if the coördinates of the vertices of a parallelogram are whole numbers, the parallelogram contains a whole number of square units. Is this theorem true for any polygon?

21. Prove that if the coördinates of the vertices of a polygon are rational numbers, the area of the polygon is also a rational number.

22. Prove that if the coördinates of the vertices of a triangle are even numbers, the area of the triangle is an even number of square units. Is this theorem true for any polygon?

11. Application of analysis to elementary geometry.
Many important theorems of elementary geometry, particularly those relating to polygons, can be proved analytically by the theory of the present chapter.

Example: Prove that the diagonals of a parallelogram bisect each other.

Let us place the parallelogram in the position shown. The coördinates of O are $(0, 0)$, those of P_1 may be taken

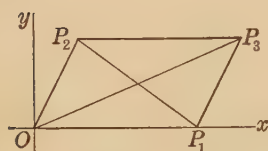


FIG. 12

as $(x_1, 0)$, those of P_2 as (x_2, y_2) , whence those of P_3 must be $(x_1 + x_2, y_2)$. Thus the midpoint of $P_1 P_2$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_2}{2}\right)$; likewise the

midpoint of OP_3 is $\left(\frac{x_1 + x_2}{2}, \frac{y_2}{2}\right)$,

which proves the theorem.

It is obvious that *if the property to be proved is independent of the position of the figure, there is no loss of generality in assuming the figure in any convenient position* with reference to the coördinate axes. Thus the proof just given is entirely general. On the other hand, the figure itself must not be made special in any way: for instance, in the above example, by assigning numerical coördinates to the vertices, or by choosing as the given parallelogram a rhombus or a rectangle.

Further, this point must be carefully noted. While in elementary geometry the proof is obtained by studying the properties of the figure, in analysis the proof arises from algebraic work involving the coördinates: the figure is of no use at all beyond helping us to bear in mind the nature of the problem. Our notation must therefore be such that *the coördinates themselves express the data* of the problem. Thus, in the above example, when coördinates have been assigned to three of the vertices, those of the fourth *are determined*, and must be correctly expressed in terms of those already assigned. If this is not done, the proof is impossible.

EXERCISES

Prove the following theorems.

1. The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

2. The diagonals of a rectangle are equal.

3. The distance between the midpoints of the non-parallel sides of a trapezoid is half the sum of the parallel sides.

4. An isosceles triangle has two equal medians.

5. A triangle having two equal medians is isosceles.

6. The diagonals of a rhombus are perpendicular.

7. If the diagonals of a rectangle are perpendicular, the rectangle is a square.

8. If a convex quadrilateral has two opposite sides equal and parallel, it is a parallelogram.

9. The lines joining the midpoints of the sides of a triangle divide it into four equal triangles.

10. A quadrilateral whose diagonals bisect each other is a parallelogram.

11. A quadrilateral whose diagonals bisect each other at right angles is a rhombus.

12. The diagonals of an isosceles trapezoid are equal.

13. A trapezoid whose diagonals are equal is isosceles.

14. The line segments joining the midpoints of opposite sides of a quadrilateral bisect each other.

15. The line segments joining the midpoints of adjacent sides of a quadrilateral form a parallelogram.

16. The sum of the squares on the sides of a parallelogram equals the sum of the squares on the diagonals.

17. The medians of a triangle intersect in a trisection point of each.

CHAPTER II

THE LOCUS OF AN EQUATION

12. Constants; variables. In algebra (and to some extent even in arithmetic) we become familiar with the use of letters to represent numerical quantities. Two kinds of quantities occur in algebra: the *knowns*, usually represented by $a, b, \dots$, and the *unknowns*, represented by $x, y, \dots$. Thus the general quadratic equation is

$$ax^2 + bx + c = 0;$$

the equation of the first degree in three unknowns is

$$ax + by + cz + d = 0.$$

In analytic geometry the quantities with which we have to deal are “constants” and “variables.”

A *constant* is a quantity whose value remains unchanged throughout any given problem. Examples are the coördinates of a fixed point, the radius of a circle, the slope of a line, etc. Coördinates of fixed points are denoted by the letters x, y with subscripts, as $(x_1, y_1), (x_2, y_2)$, etc.; other letters, such as a, b, m , etc., are also used to denote constants.

A *variable* is a quantity that may take different values (usually an infinite number of them) in the same problem. The variables most frequently occurring are the coördinates (x, y) of a point moving along a definite path.

For example, the fact that the point (x, y) is at the constant distance a from the origin is expressed by the equation

$$x^2 + y^2 = a^2.$$

This equation evidently remains true if the point (x, y) moves along the circle of radius a with center at O .

13. The locus of an equation. Let two variables x and y be connected by an equation—for instance,

$$y = \frac{1}{2}x + 2, \quad y^3 = 2x, \text{ etc.}$$

If any value be assigned to either variable, one or more values of the other are determined, in general, by the equation; thus there exist infinitely many pairs of values of x and y that satisfy the equation. Each pair of numbers may be represented geometrically by a point. The points so determined are not scattered at random throughout the plane, but form in the aggregate a definite *curve*. This curve is called the *locus* of the equation:

The locus of an equation is a curve containing those points, and only those points, whose coördinates satisfy the equation.

The curve corresponding to a given equation is said to *represent the equation geometrically*, while the equation *represents the curve analytically*. The study of plane curves by means of the equations representing them forms the subject-matter of plane analytic geometry.

The elementary method of tracing a curve is to find a number of pairs of values of x and y satisfying the equation, plot the corresponding points, and draw a smooth curve through them: this curve is approximately the desired locus. The great objection to this method is that it sheds but little light on the general properties of the curve. Better methods will be developed as we proceed.

It should be remarked that in exceptional cases an equation may represent only a single point, or it may have no locus whatever. For example, the equation $x^2 + y^2 = 0$ is evidently satisfied only by the coördinates $(0, 0)$; the equation $x^2 + y^2 = -1$ is satisfied by no (real) values of x and y . Such forms are of little importance.

To plot a curve by points, we proceed as in the examples below. Occasionally it is convenient to adopt different scales in the two directions (see § 4). The scale or scales adopted should always be noted on the drawing.

Examples: (a) Trace the curve $y = \frac{1}{2}x + 2$.

We assign values to x and compute the values of y :

x	0	1	2	-1	-2	-3	-4	-5
y	2	$\frac{5}{2}$	3	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$

We now plot these points, choosing say two spaces on the coördinate paper as the unit, and draw a smooth curve through them. It should be noted that the curve picks up the points *according to the algebraic order of the values of x* —not as they are arranged in the above table. The “curve” in this case appears to be a straight line: it will be proved in § 27, and assumed meanwhile, that *the locus of every equation of the first degree is a straight line*.

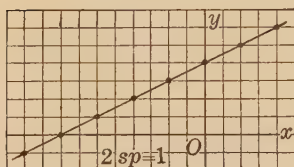


FIG. 13

(b) Plot the curve

$$y^3 = 2x.$$

Here, to avoid the extraction of cube roots, let us write the equation in the form

$$x = \frac{1}{2}y^3$$

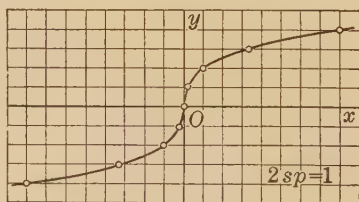


FIG. 14

and assign values to y . Since x becomes very large as y increases, we will assign to y only small values:

y	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2
x	0	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{27}{16}$	4	$-\frac{1}{16}$	$-\frac{1}{8}$	$-\frac{27}{16}$	-4

14. Intercepts on the axes. To find the points where the curve crosses Ox we must evidently *put $y = 0$ and solve for x* ; to find the intersections with Oy we *put $x = 0$ and solve for y* . Of course in some cases the solution cannot be carried out on account of algebraic difficulties.

The distances cut off by a curve on the axes are called its *x -* and *y -intercepts*.

Example: The line $3x + 2y = 12$ has intercepts 4 on the x -axis and 6 on the y -axis: i.e. it passes through the points (4, 0) and (0, 6).

15. Symmetry. Two points P_1, P_2 are said to be *placed symmetrically*, or to be *symmetric*, with respect to a line, if that line is the perpendicular bisector of the segment P_1P_2 ; the line is then called an *axis of symmetry* (the line λ in Fig. 15). Each point is then said to be the *image*, or *reflection*, of the other in the line λ . A curve or other plane figure is said to be *symmetric with respect to an axis* * if, corresponding to every point P_1 of the figure, the image-point P_2 in that axis also belongs to the figure. This means

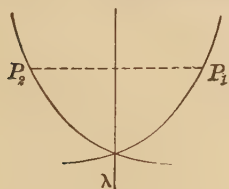


FIG. 15

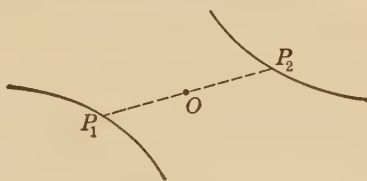


FIG. 16

that the figure is unchanged by reflection in the axis—i.e. that the part of the figure on one side of the axis is the image of the part on the other side. Figure 15 shows a curve symmetric with respect to the indicated line.

* Obviously, this use of the word “axis” has nothing to do with “axis” of coördinates. There are always just two axes of coördinates, whereas a curve may have no axis of symmetry, or one, or more; further, if there is an axis of symmetry, this line need not be chosen as an axis of coördinates.

Two points P_1, P_2 are said to be placed *symmetrically with respect to a point O* if O is the midpoint of P_1P_2 . A plane figure is symmetric with respect to a point O if, corresponding to every point P_1 of the figure, there is a point P_2 , also belonging to the figure, such that O is the midpoint of P_1P_2 . The point O is called the *center of symmetry*, or simply the *center*. In Fig. 16, the curve has the point O as center of symmetry.

16. Tests for symmetry. As direct consequences of the definitions of § 15 we deduce analytic tests for symmetry of a curve with respect to the coördinate axes and the origin.

THEOREM I: *A curve is symmetric with respect to the x -axis if its equation is unchanged* when y is replaced by $-y$; and conversely. Similarly for symmetry with respect to the y -axis.*

THEOREM II: *A curve is symmetric with respect to the origin if its equation is unchanged when x is replaced by $-x$ and y by $-y$ simultaneously; and conversely.*

Proof of I: By hypothesis, if any pair of coördinates (x, y) satisfy the equation, the coördinates $(x, -y)$ of the image-point with respect to the x -axis also satisfy the equation. The proof of the converse is left to the student.

The proof of II is also left as an exercise.

Example: Trace the curve

$$x^2 - y^2 = 4.$$

When $y = 0$, $x = \pm 2$; when $x = 0$, y is imaginary, so that the curve does not intersect Oy . The curve is symmetric to both axes.

* More precisely, if the new form of the equation is *equivalent* to the original form—i.e. if each form is satisfied by all the values of the unknowns that satisfy the other.

Writing the equation in the form

$$y = \pm \sqrt{x^2 - 4},$$

we see that y is imaginary if x is numerically less than 2. Values of x greater than 2 give the following points:

x	3	4	5	6
y	$\pm\sqrt{5}$	$\pm 2\sqrt{3}$	$\pm\sqrt{21}$	$\pm 4\sqrt{2}$

On account of the symmetry with respect to Oy , the portion of the curve corresponding to negative values of x can be obtained by reflection.

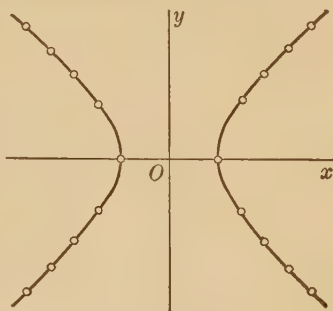


FIG. 17

EXERCISES

Draw the following straight lines. Plot three points in each case—the third as a check.

1. $2x - y = 6.$

2. $x - 3y = 3.$

3. $3x - 5y + 6 = 0.$

4. $5x - 4y + 1 = 0.$

5. $6x + 5y + 1 = 0.$

6. $x + 2y + 30 = 0.$

7. $3x - 5y = 0.$

8. $2x + 3y = 0.$

9. $x + 2 = 0.$

10. $y = 5.$

Find the intercepts on the axes, test for symmetry, plot a number of points, and trace the curve, choosing a suitable scale in each case.

11. $y = 2x^2 + 3.$

12. $y = 1 - 4x^2.$

13. $y = x^2 - 3x.$

14. $y = 3 + 2x - x^2.$

15. $x + y + y^2 = 0.$

16. $x - 2y^2 - 4y = 0.$

17. $x^2 + 2y^2 = 2.$

18. $x^2 + 3y^2 = 9.$

19. $9x^2 + 4y^2 = 1.$

20. $4x^2 + y^2 = 400.$

21. $4x^2 - y^2 = 4.$

22. $x^2 - 2y^2 = 2.$

23. $x^2 - y^2 + 1 = 0.$

24. $x^2 - 4y^2 + 16 = 0.$

25. $y = x^3 - 5x^2 + 6x.$

26. $y = x^3 + 7x^2 + 6x.$

27. $y = x^3 - 2x^2 - 9x + 18.$

28. $y = -x^3 + 3x^2 - 4.$

29. $x = (y^2 - 1)^2.$

30. $x = y^2(4 - y^2).$

Without formal proofs, state how many axes of symmetry are possessed by each of the following figures.

31. (a) A circle; (b) a circular arc.
 32. (a) A straight line; (b) a straight line segment. *Ans. (b) Two.*
 33. A triangle. Discuss special cases fully.
 34. A quadrilateral. Discuss all special cases.
 35. A regular hexagon.
 36. Show that two circles taken together always have one axis of symmetry. When are there two such axes? When more than two?
 37. When do three circles have one or more axes of symmetry?
 38. When do two circles have a center of symmetry?
 39. When do three circles have a center of symmetry?
 40. Prove analytically that if a curve is symmetric with respect to Ox and Oy , it is symmetric with respect to the origin. Show by an example that the converse is not true.
 41. Prove the theorem of Ex. 40 geometrically.
 42. Prove that if a curve is symmetric to one axis and the origin, it is symmetric to the other axis also.

17. Consequences of the definition of locus. From the definition of the locus of an equation (§ 13) we may deduce certain rules which are so important that, in spite of their obviousness, it seems worth while to state them explicitly.

By the definition, a point lies on a curve *if and only if its coördinates satisfy the equation of the curve*. This suggests

RULE I: *To find out whether a point lies on a given curve, substitute its coördinates for x and y in the equation of the curve.*

Example: (a) The point $(2, 12)$ lies on the curve

$$y = 3x^2$$

because

$$12 = 3 \cdot 4;$$

the point $(-1, -3)$ does not lie on the curve because

$$-3 \neq 3 \cdot 1.$$

RULE II: *To express analytically the condition that a point shall lie on a curve, write the equation of the curve with the co-ordinates of the point substituted for x and y .*

Examples: (b) The point (x_1, y_1) lies on the curve

$$y^2 = 4x$$

if and only if

$$y_1^2 = 4x_1.$$

(c) Determine the constant a so that the point $(-2, -3)$ shall lie on the curve

$$y = ax^2.$$

Substituting the coördinates $(-2, -3)$, we get

$$-3 = 4a, \quad \text{or} \quad a = -\frac{3}{4},$$

and the equation of the curve is

$$y = -\frac{3}{4}x^2.$$

RULE III: *To find the ordinate of a point on a curve, being given its abscissa, substitute the given abscissa for x in the equation of the curve and solve for y . Similarly we may find the abscissa when the ordinate is given.*

18. Factorable equations. In algebra, when the product of two or more factors is equated to 0, the equation thus formed is satisfied whenever any one of the factors is 0. For instance, the equation

$$(1) \quad 3(x + 2y)(x^2 - y^2) = 0$$

is satisfied if and only if

$$(2) \quad x + 2y = 0,$$

$$(3) \quad x + y = 0,$$

or

$$(4) \quad x - y = 0.$$

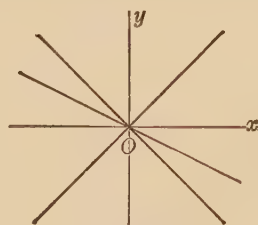


FIG. 18

Hence, the locus of an equation whose right member is 0 and whose left member can be factored consists of all points whose coördinates when substituted in the equation cause any one of the factors to vanish. That is, the curve represented by such an equation breaks up into several distinct curves, viz. the loci of the equations formed by equating the several factors separately to 0. Thus equation (1) above represents (Fig. 18) the three straight lines (2), (3), (4).

19. Transformation of equations. When an algebraic operation of any kind is performed upon an equation, the locus is evidently unchanged provided that, after the transformation, the equation is satisfied by all those pairs of values of x and y , and only those, which satisfied it originally. This is the case with such operations as addition of equals to each member, multiplication of both members by a constant factor, etc. However, the locus is in general changed by introducing or striking out a variable factor, squaring both members, etc. For instance, the equation

$$y = x$$

represents a certain straight line; the equation

$$y^2 = x^2$$

holds not only when $y = x$, but also when $y = -x$, and thus represents two straight lines.

20. Points of intersection of two curves. The points of intersection of two curves are points whose coördinates *satisfy both equations*, and there are no other points having this property. Hence *the points of intersection of two curves are found by solving the equations of the curves as simultaneous equations.*

It may happen that all the values of x and y , found by solving two simultaneous equations, are imaginary; or the equations may be “incompatible”—i.e. not satisfied by any pairs of values either real or imaginary (e.g. Ex. 28 below). Obviously, in either case the result means that the curves do not intersect.

After solving two simultaneous equations the results should always be tested by substituting the values of x and y in both equations and noting whether the equations hold.

Example: Find the points of intersection of the straight line

$$(1) \quad 2x + y = 10$$

and the circle

$$(2) \quad x^2 + y^2 = 25.$$

Substituting the value of y from (1) in (2), we get

$$x^2 + (10 - 2x)^2 = 25,$$

or

$$5x^2 - 40x + 75 = 0,$$

$$x^2 - 8x + 15 = 0,$$

whence

$$x = 3 \text{ or } 5.$$

By (1),

$$y = 4 \text{ or } 0,$$

and the points are (3, 4), (5, 0).

Check:

$$6 + 4 = 10, \quad 9 + 16 = 25;$$

$$10 + 0 = 10, \quad 25 + 0 = 25.$$

The curves are shown in the figure.

In algebra, the *degree* of a term involving x and y is the sum of the exponents of x and y in the term. Thus the

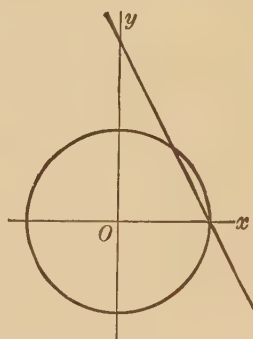


FIG. 19

terms x^2 , $3xy$, $2y^2$ are of the second degree in x and y ; $3x^4$, $2xy^3$ are of the fourth degree.

The *degree* of an algebraic equation in x and y is that of the term of highest degree when the equation is rationalized and cleared of fractions.* Thus the equation

$$(3) \quad \sqrt{3}x + \frac{2}{3}y = 5$$

is of the first degree; the equation

$$y = \frac{1}{x^2}$$

is of the third degree; the equation

$$y = x^{2/3}$$

is of the third degree.

A curve whose equation is of the n th degree is spoken of briefly as a *curve of the n th degree*.

It is shown in algebra that when two equations in x and y are solved as simultaneous, the number of solutions (i.e. the number of pairs of values of x and y satisfying both equations) is not greater than the product of the degrees of the equations. Hence *the number of points of intersection of two curves is not greater than the product of the degrees of their equations*.

EXERCISES

Determine whether the given points lie on the given curve.

1. Curve $4x + y = 6$; points $(1, 2)$, $(2, -1)$, $(4, -10)$, $(0, -6)$.
2. Curve $2x - 3y + 4 = 0$; points $(1, 3)$, $(-2, 0)$, $(7, 6)$, $(10, 8)$.
3. Curve $y = x^3 - 4x^2 + 2x + 5$; points $(-1, 6)$, $(2, 1)$, $(3, 2)$.
4. Curve $x^2 - xy + 2y^2 + 6y = 4$; points $(2, -2)$, $(-2, 0)$, $(4, 2)$.
5. Curve $y^2 = 4ax$; points $(2a, a)$, $(-a, -2a)$, $(4a, -4a)$, $(\frac{1}{4}a, a)$.
6. Curve $x^2 + y^2 = 5$; points $(-2, -1)$, $(\sqrt{2}, \sqrt{3})$, $(1.41, 1.73)$.

* As regards the variables only: irrational and fractional constants may be present, as in (3).

7. Determine k so that the straight line $x + 2y = k$ shall pass (a) through $(4, 1)$; (b) through $(-6, -4)$; (c) through $(0, 0)$.

8. For what value of a does the curve $y = ax^3$ pass through $(-1, -3)$? Through $(4, 2)$? Through $(0, 0)$? Through $(0, 2)$? Through $(2, 0)$?

9. Determine m so that the line $y = mx + 2$ shall pass (a) through $(1, 3)$; (b) through $(2, -2)$; (c) through $(0, 2)$; (d) through $(0, 0)$.

10. What is the condition that the curve $y = ax^2 + bx + c$ shall pass through $(0, 0)$? Through $(2, 1)$? Through $(-1, c)$?

11. Given that the point (x_1, y_1) lies on the curve $x^2 + 4y^2 = 9$, express this fact analytically.

12. In what case does the point (h, k) lie on the line $y = 3x + 5$?

13. Determine A and B so that the line $Ax + By = 1$ shall pass through the points $(3, -2)$, $(4, 2)$.

14. Determine A and B so that the line $Ax + By = 1$ shall pass through the points $(7, 2)$, $(1, -4)$.

15. Determine a and b so that the curve $x^2 + y^2 + ax + by = 0$ shall pass through the points $(1, 2)$, $(-3, 3)$. Ans. $a = \frac{7}{3}$, $b = -\frac{1}{3}$.

16. On the curve $y^2 = x^3$, find (a) the point whose ordinate is -1 ; (b) the points whose abscissa is 4 .

17. On the curve $y^2 - x - 3y + 2 = 0$, find (a) the point whose ordinate is 3 ; (b) the points whose abscissa is 0 ; (c) the points whose abscissa is 3 ; (d) the points whose abscissa is -4 .

18. On the curve $y^2 = x^3 - 2x^2 - 5x + 7$, find the points (a) whose abscissa is -1 ; (b) whose ordinate is -1 ; (c) whose abscissa is 2 .

Ans. (c) None.

Trace the following curves, after factoring the equations.

19. $x^2 + 3xy = 0$.

20. $x^3y = 4xy^3$.

21. $x^2 - 4xy + 4y^2 = 0$.

22. $x^2 - 4xy + 4y^2 = 4$.

23. $xy^2 = x^2$.

24. $y^2 = x^4$.

In the following cases, (a) find the points of intersection of the given curves; (b) check by substitution in the given equations; (c) plot the two curves on the same axes.

25. $4x + 4y = 5$, $x - 2y + 1 = 0$.

26. $3x + 6y + 1 = 0$, $x - 5y = 2$.

27. $y = 2x - 3$, $4x - 2y = 3$.

28. $3x - y = 0$, $2y - 6x + 3 = 0$.

29. $x^2 + y^2 = 5$, $3x - y = 5$.

30. $x^2 + y^2 = 10$, $2x + 3y = 9$.

31. $x^2 + y^2 = 10, \quad 3x + y = 10.$

32. $x^2 + y^2 = 17, \quad x + 4y = 17.$

33. $x^2 + y^2 = 5, \quad x^2 = 2y + 5.$

34. $y = x^3 - 6x^2 + 4x + 2, \quad 8x + y = 10.$

Ans. (2, -6) three times.

35. $y = x^3 - 2x^2 - 5, \quad x + y + 5 = 0.$

36. $y = x^3, \quad x + y + 10 = 0.$

37. $y = 2x^2 - 3x + 1, \quad y = x^3 - 4x^2 + 2x + 1.$

38. $4y = -4x^3 - 3x^2 + 14x + 8, \quad 4y = 4x^3 - 3x^2 - 18x + 8.$

39. $y = x^2, \quad 2x = 3y - y^2. \quad \text{Ans. } (0, 0), (1, 1), (1, 1), (-2, 4).$

40. $y = x^2, \quad y^2 + 6x - 7y = 0. \quad \text{Ans. } (0, 0), (1, 1), (2, 4), (-3, 9).$

Determine the degree of each of the following equations.

41. $(x - 1)(y + 2) = 3.$

42. $x(xy - 3) = x^2 + y^2.$

43. $\sqrt{3}(x - y) = 4.$

44. $y = \pm \sqrt{3x}.$

45. $y = \frac{1}{x + 1}.$

46. $y = \frac{x}{x^2 + 1}.$

47. Show that the equation $x^{1/2} \pm y^{1/2} = \pm a^{1/2}$ is of the second degree.48. Show that the equation $x^{2/3} + y^{2/3} = a^{2/3}$ is of the sixth degree.

CHAPTER III

THE EQUATION OF A LOCUS

21. Path of a moving point. The work of Chapter I illustrated in a preliminary way the basic reason for the existence of analytic geometry—viz. that the introduction of a coördinate system makes it possible to translate geometric problems into algebraic language, and to solve them by purely algebraic methods.

In Chapter II we took a long step in advance, when it appeared that a *plane curve* may be represented algebraically by an *equation* in x and y , whence the geometric properties of the curve may be discovered by merely interpreting geometrically the algebraic properties of the equation. In that chapter, however, the equation of the curve was always given in advance. Unfortunately, in practice the equation is very seldom given to begin with; it is usually necessary, as a first step, to *derive* the equation from a geometric definition of some kind. This problem will occupy us in the present chapter.

Very often the curve is defined as the *path*, or *locus*, of a *point which moves according to a given law*. In such a case, the statement of the law of motion usually suggests an equality between certain distances or other quantities involving the coördinates of the moving point. We always *denote the coördinates of the moving point* by (x, y) , and try to obtain (by the formulas of analytic geometry) expressions for the distances or other quantities involved in the statement of the law of motion, after which it is usually a simple matter to write out the equation of the locus.

Examples: (a) A point moves so as to remain always equidistant from the points $P_1 : (3, 2)$ and $P_2 : (-1, 5)$. Find the equation of its locus.

The law of motion states that the distance from the moving point $P : (x, y)$ to the point $(3, 2)$ equals the distance from (x, y) to $(-1, 5)$. Hence, by § 5,

$$\sqrt{(x-3)^2 + (y-2)^2} = \sqrt{(x+1)^2 + (y-5)^2}.$$

This equation expresses the given law of motion and is therefore *the equation of the locus*.* Squaring and expanding, we get

$$\begin{aligned} x^2 - 6x + 9 + y^2 - 4y + 4 \\ = x^2 + 2x + 1 + y^2 - 10y + 25, \end{aligned}$$

or

$$8x - 6y + 13 = 0,$$

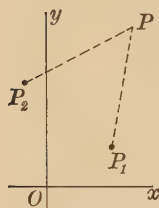


FIG. 20

which shows that the locus is a straight line.

It is of course the perpendicular bisector of the line joining the fixed points.

(b) A point moves so as to remain equidistant from the line $y = -1$ and the point $P_1 : (0, 1)$. Find the equation of its locus.

The distance of any point $P : (x, y)$ from the line $y = -1$ is $y + 1$, as is evident from the figure. Hence we have

$$\begin{aligned} y + 1 &= \sqrt{x^2 + (y-1)^2}, \\ y^2 + 2y + 1 &= x^2 + y^2 - 2y + 1, \\ x^2 &= 4y. \end{aligned}$$

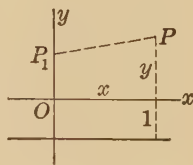


FIG. 21

* To prove strictly that a curve and its equation as found in the present chapter correspond to each other in the manner required by the definition of "locus of an equation" (§ 13), it is clearly necessary to show (a) that the coördinates of every point of the locus satisfy the equation; (b) that every point whose coördinates satisfy the equation lies on the locus. The method outlined above establishes only the first of these statements. However, in most cases the second part of the proof can be carried out by merely reversing the steps already taken, and therefore will usually be omitted.

EXERCISES

In each of the following cases, find the equation of the locus.

1. A moving point is always equidistant from the points (3, 1), (-2, 5). *Ans.* $8y = 10x + 19$.

2. A moving point is always equidistant from (-2, 2), (5, -3).

3. A point moves so that its distance from (4, 0) is always twice its distance from (1, 0). Draw the curve. *Ans.* $x^2 + y^2 = 4$.

4. A point moves so that its distance from (0, 4) is two-thirds of its distance from (0, 9). Draw the curve. *Ans.* $x^2 + y^2 = 36$.

5. A moving point is always at the distance 4 from (1, -2).

6. A moving point is always equidistant from the line $y = 3$ and the point (0, -3). *Ans.* $x^2 + 12y = 0$.

7. A moving point is always equidistant from the y -axis and the point (4, 0). *Ans.* $y^2 = 8x - 16$.

8. A moving point is always equidistant from the line $x + 3 = 0$ and the point (2, 1). *Ans.* $y^2 - 10x - 2y = 4$.

9. A moving point is always equidistant from the line $y + 4 = 0$ and the point (3, -2). *Ans.* $x^2 - 6x - 4y = 3$.

10. A point moves so that its distance from the point (0, 9) is three times its distance from the line $y = 1$. *Ans.* $8y^2 - x^2 = 72$.

11. A point moves so that its distance from the point (1, 0) is one-half of its distance from the line $x = 4$. *Ans.* $3x^2 + 4y^2 = 12$.

12. A point moves so that its distance from (0, 5) is two-thirds of its distance from the x -axis. *Ans.* $9x^2 + 5y^2 - 90y + 225 = 0$.

13. A point moves so that its distance from (3, 2) is twice its distance from the y -axis. *Ans.* $3x^2 - y^2 + 6x + 4y = 13$.

14. A point moves so that the sum of the squares of its distances from (1, 0), (-1, 0) is 10. Draw the curve. *Ans.* $x^2 + y^2 = 4$.

15. A point moves so that the sum of the squares of its distances from (0, 3), (0, -3) is 20. Draw the curve. *Ans.* $x^2 + y^2 = 1$.

16. A point moves so that the sum of its distances from (4, 0) and (-4, 0) is 10. *Ans.* $9x^2 + 25y^2 = 225$.

17. A point moves so that the sum of its distances from (0, 1) and (0, -1) is $2\sqrt{2}$. *Ans.* $2x^2 + y^2 = 2$.

18. A point moves so that the difference of its distances from (2, 5) and (6, 5) is 2. *Ans.* $3x^2 - y^2 - 24x + 10y + 20 = 0$.

19. A point moves so that the difference of its distances from (0, 0) and (0, 8) is 4.

20. Two vertices of a triangle are (2, 1), (4, 3); the area is 2. Find the locus of the third vertex. *Ans.* $x - y = 3$; $x - y + 1 = 0$.

21. Two vertices of a triangle are (3, 5), (1, 4); the area is 1. Find the locus of the third vertex. *Ans.* $x - 2y + 5 = 0$; $x - 2y + 9 = 0$.

22. The base of a triangle joins (4, 3), (0, 5); the altitude is 2. Find the locus of the third vertex. *Ans.* $x + 2y = 10 \pm 2\sqrt{5}$.

23. The base of a triangle joins (2, 5), (-1, 3); the altitude is 1. Find the locus of the third vertex. *Ans.* $2x - 3y + 11 = \pm\sqrt{13}$.

24. Two vertices of a right triangle are (1, 1), (4, 3), with the right angle at (1, 1). Find the locus of the third vertex by using § 9.

25. Solve Ex. 24 by using § 5.

26. Solve Ex. 24 by using § 7.

27. Two vertices of a right triangle are (0, 6), (5, 2), with the right angle at (0, 6). Find the locus of the third vertex in three ways.

28. The hypotenuse of a right triangle joins the points (5, -1), (3, 3). Find the locus of the third vertex in three ways. (§§ 5, 7, 9.)

29. The hypotenuse of a right triangle joins the points (0, 0), (8, -6). Find the locus of the third vertex in three ways.

30. Solve Ex. 24 by using § 10.

31. The hypotenuse of a right triangle joins the points (1, 0), (-1, 0). Find the locus of the third vertex by using § 10.

22. Loci defined geometrically. Sometimes we have a curve actually drawn, and are required to find its equation from the known geometric properties of the figure, as in example (a) below; or a geometric construction may be given by means of which the points of the curve are determined, as in (b).

In such cases we *assume a point of the curve in a general position, and denote its coördinates by* (x, y) . Then the problem is merely to express some characteristic property of the curve by means of an equation involving x , y , and the constants of the problem. (By “characteristic property” is meant, of course, one that holds for all the points of the curve, and for no other points.) No matter what property is used, the result must be the equation of the curve.

Examples: (a) Find the equation of the straight line whose intercepts on the axes are $OA = 3$ and $OB = 2$ respectively.

Assume a point $P : (x, y)$ in a general position on the line. It is clear that the triangles MAP and OAB are similar if and only if P is on the line. Hence if the fact that these triangles are similar be expressed by an equation involving x and y , that equation must represent the line. Now

$$\frac{MP}{OB} = \frac{MA}{OA};$$

that is,

$$\frac{y}{2} = \frac{3 - x}{3}.$$

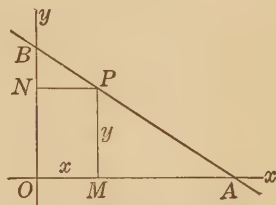


FIG. 22

Clearing of fractions and transposing, we get the form

$$2x + 3y = 6.$$

Check: When $y = 0$, $x = 3$; when $x = 0$, $y = 2$.

(b) Find the locus of the centers of circles passing through the point $P_1 : (0, 1)$ and tangent to the line $y + 1 = 0$.

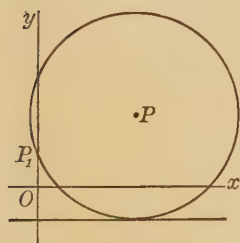


FIG. 23

Let us draw one* of the circles in question, and let its center P be the point (x, y) . We note that P is equidistant from the line $y + 1 = 0$ and the point $(0, 1)$, so that the locus is the same as in example (b), § 21:

$$y + 1 = \sqrt{x^2 + (y - 1)^2},$$

$$x^2 = 4y.$$

* To draw more than one would merely obscure the figure, without helping us to find the equation of the locus.

EXERCISES

1. Find the equation of the perpendicular bisector of the line joining $(4, 1)$, $(5, -3)$.

2. Find the equation of the perpendicular bisector of the line joining $(0, 5)$, $(4, -3)$. *Ans.* $2y = x$.

3. Find the locus of the centers of circles passing through the points $(1, 5)$, $(-3, -1)$. *Ans.* $2x + 3y = 4$.

4. Find the locus of the centers of circles passing through the points $(-1, -3)$, $(3, -1)$. *Ans.* $2x + y = 0$.

5. The base of an isosceles triangle is the line from $(7, 3)$ to $(-7, 1)$. Find the locus of the third vertex by two methods.

6. The base of an isosceles triangle joins the points $(0, 4)$, $(-6, 0)$. Find the locus of the third vertex by two methods.

7. Solve example (a) by making use of the fact that the triangles *MAP* and *NPB* are similar.

8. Solve example (a) by means of the fact that the area of the triangle *APB* is 0. (§ 10.)

9. By two methods, find the equation of the straight line through $(4, -1)$, $(-6, 3)$.

10. By two methods, find the equation of the straight line through $(-5, 2)$, $(4, 6)$.

11. Find the equation of a circle of radius 4 with center at $(2, -3)$.

12. Find the equation of a circle of radius 2 with center at $(-5, 1)$.

13. A circle is drawn with center $(3, 1)$ and passing through $(1, 3)$. Find its equation.

14. A circle is drawn with center $(-2, 0)$ and passing through $(6, 1)$. Find its equation.

15. A circle is drawn having the points $(1, 1)$, $(5, -1)$ as ends of a diameter. Find its equation by three methods.

16. A circle is drawn having the points $(4, 1)$, $(2, 3)$ as ends of a diameter. Find its equation by three methods.

17. Find the locus of the centers of circles passing through $(2, 5)$ and touching the line $x + 2 = 0$.

18. Find the locus of the centers of circles passing through $(-3, -5)$ and touching the line $y + 3 = 0$.

19. One of the equal sides of an isosceles triangle is the line from $(4, 2)$ to $(3, 3)$. Find the locus of the third vertex.

Ans. $x^2 + y^2 - 6x - 6y + 16 = 0$; $x^2 + y^2 - 8x - 4y + 18 = 0$.

20. A variable circle is tangent externally to two fixed circles of radius 1 with centers at (5, 0) and (1, 7). Find the locus of the center of the variable circle. *Ans.* $8x - 14y + 25 = 0$.

21. A moving circle is tangent to the y -axis and to a circle of radius 1 with center at (2, 0). Find the locus of the center of the moving circle. Draw the curve. *Ans.* $y^2 - 6x + 3 = 0$; $y^2 - 2x + 3 = 0$.

22. A moving circle is tangent to the x -axis and to a circle of radius 2 with center at (0, 4). Find the locus of the center of the moving circle. Draw the curve.

23. Find the equation of the line parallel to and at a distance 2 from the line joining (4, 3), (0, 6). (§ 10.) *Ans.* $3x + 4y = 24 \pm 10$.

24. A circle of radius 3 rolls on the line through (3, -5), (-1, -6). Find the locus of its center. (§ 10.) *Ans.* $x - 4y = 23 \pm 3\sqrt{17}$.

25. A circle of radius $\sqrt{5}$ rolls on the line through (2, 0), (0, -4). Find the locus of its center. *Ans.* $2x - y = 4 \pm 5$.

26. Two circles of radius 3 are drawn with centers at (0, 0), (3, 2). Find the equations of their common tangents. *Ans.* $2x - 3y = \pm 3\sqrt{13}$.

27. Two circles of radius 5 are drawn with centers at (6, 0), (0, -8). Find the equations of their common tangents.

$$\text{Ans. } 4x - 3y = 24 \pm 25; 3x + 4y + 7 = 0.$$

28. Two circles are drawn: one of radius $\sqrt{5}$ with center at (1, 3), the other of radius $2\sqrt{5}$ with center at (0, 1). Find the equation of their common tangent. (§ 6.) *Ans.* $x + 2y = 12$.

29. A line segment of length $2a$ moves with its ends in the coordinate axes. Find the locus of its midpoint. *Ans.* $x^2 + y^2 = a^2$.

30. A line segment of length $3a$ moves with its ends in the coordinate axes. Find the locus of each of its points of trisection. Draw the curves. *Ans.* $x^2 + 4y^2 = 4a^2$; $4x^2 + y^2 = 4a^2$.

31. Two circles of radius 10 are drawn with centers at (7, 1) and (-7, 3). Find the equation of their common chord, and the length of the chord. *Ans.* $y = 7x + 2$; $10\sqrt{2}$.

32. Two circles of radius 7 are drawn with centers at (3, 2), (-5, -4). Find the equation of their common chord, and the length of the chord.

CHAPTER IV

THE STRAIGHT LINE

23. Line parallel to a coördinate axis. We proceed now to study various special curves, beginning with the straight line.

We note first that if a straight line is parallel to the y -axis, its equation is

$$x = k,$$



FIG. 24

where k is the distance of the line from the axis; and conversely. For all points on that line, and no other points, have the abscissa k .

Similarly, a line parallel to and at a distance k from the x -axis has the equation

$$y = k.$$

Figure 24 shows the lines $x = 2$, $y = -3$.

24. Line through a given point in a given direction: point-slope form. Let us try to find the equation of a straight line passing through a given point $P_1 : (x_1, y_1)$ in a given direction—i.e. having a given slope $m = \tan \alpha$.

Assuming a point $P : (x, y)$ in a general position on the line (cf. § 22), we note that, in the triangle P_1MP ,

$$\tan \alpha = \frac{MP}{P_1M} = m;$$

that is,

$$\frac{y - y_1}{x - x_1} = m,$$

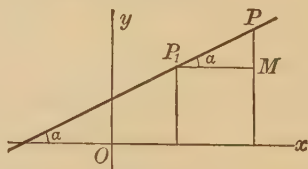


FIG. 25

or

$$(1) \quad y - y_1 = m(x - x_1).$$

This is the *point-slope form* of the equation of the straight line. It is one of several so-called *standard forms* of the equation of the straight line which will be developed in this chapter.

The point-slope form fails in case the line is parallel to the y -axis, since for such a line $m = \tan 90^\circ$, which is non-existent. But by § 23, the equation of a line through (x_1, y_1) parallel to Oy is simply

$$x = x_1.$$

Example: Find the equation of the line through $(3, -6)$ perpendicular to the line joining $(4, 1)$ and $(2, 5)$.

The slope of the line joining $(4, 1)$ and $(2, 5)$ is -2 , whence the slope of the required line is $\frac{1}{2}$, and by (1) its equation is *

$$y + 6 = \frac{1}{2}(x - 3),$$

or

$$x - 2y = 15.$$

When the slope and one point of a line are given, the line can evidently be drawn, if desired, *without writing the equation*. Thus, to draw the line of slope $\frac{3}{2}$ through the point $(4, 2)$: starting at $(4, 2)$, we measure off 2 units to the right and then 3 upward (or 4 to the right and 6 upward, or 2 to the left and 3 downward, etc.); through the point thus reached and $(4, 2)$ we draw the line.

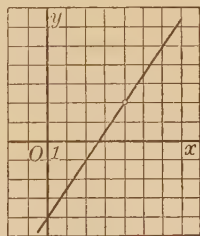


FIG. 26

* It is hardly necessary to remark that this example could be solved, and in fact similar problems have been solved, by the fundamental process of Chapter III (e.g. Exs. 24–27, p. 32). In establishing formula (1) we have merely generalized the process once for all, and in future will use the formula as a time-saver. The same remark applies to other formulas later.

25. Line through two points. By means of the point-slope form the equation of the straight line through two given points may be written down at once (unless the two points have the same abscissa, in which case the line is parallel to Oy). For we can find the slope of the line by formula (2), § 8, and can then use this value of m and the coördinates of either of the given points in (1), § 24.

Example: Find the equation of the line through the points $(3, -5)$, $(-6, -2)$.

In this case

$$m = \frac{-2 + 5}{-6 - 3} = -\frac{1}{3},$$

whence the equation of the line is

$$y + 5 = -\frac{1}{3}(x - 3),$$

or

$$x + 3y + 12 = 0.$$

$$\text{Check: } 3 - 15 + 12 = 0, \quad -6 - 6 + 12 = 0.$$

Second method: Since three points lying in a straight line form a triangle of area 0, the equation is (§ 10)

$$\begin{vmatrix} x & y & 1 \\ 3 & -5 & 1 \\ -6 & -2 & 1 \end{vmatrix} = 0, \text{ etc.}$$

EXERCISES

Draw the following lines.

1. $x = 0$.

2. $y = 0$.

3. $y + 4 = 0$.

4. $y = 5$.

5. $x - 6 = 0$.

6. $x + 7 = 0$.

Without writing the equations, draw the following lines.

7. Of slope $\frac{2}{3}$ through $(4, 1)$.

8. Of slope $\frac{5}{2}$ through $(1, -3)$.

9. Of slope $-\frac{5}{7}$ through $(-1, -2)$.

10. Of slope $-\frac{1}{3}$ through $(3, -5)$.

Write the equations of the following lines, using formula (1) of § 24.

11. Through (1, 5) (a) parallel, (b) perpendicular to the line through (4, -1), (3, 2).

12. Through (3, -4) (a) parallel, (b) perpendicular to the line through (0, -5), (4, -3).

13. Through (5, 0) (a) parallel, (b) perpendicular to the line through (-2, -4), (-3, -2).

14. Through (-3, 6) (a) parallel, (b) perpendicular to the line through (-3, -4), (5, -1).

Write the equation of the line by two methods. Check the answer.

15. Through (1, 0), (4, 2).

16. Through (5, 3), (-1, 5).

17. Through (-1, -2), (-3, 5).

18. Through (2, -3), (-3, -4).

19. Write the equations of the lines through (4, 2) (a) parallel, (b) perpendicular to the line through (3, 5), (3, 3).

20. Write the equations of the lines through (0, 5) (a) parallel, (b) perpendicular to the line through (-3, 2), (4, 2).

21. Find the equations of the medians of the triangle whose vertices are (5, 3), (7, -1), (3, -3). Find their point of intersection by two methods (§§ 6, 20). Ans. (5, - $\frac{1}{3}$).

22. In the triangle with vertices (2, 0), (3, 2), (4, -3), find the equations of the altitudes, and their point of intersection. Ans. (- $\frac{9}{2}$, - $\frac{4}{3}$).

23. Write the equation of the line of slope 2 through (5, 1). Does this line pass through (20, 32)? Through (-2, -13)?

26. Slope form. Given a line of slope m whose y -intercept is b , let us assume a point $P : (x, y)$ on the line. Then

$$m = \frac{MP}{QM} = \frac{y - b}{x},$$

whence

$$(1) \quad y = mx + b.$$

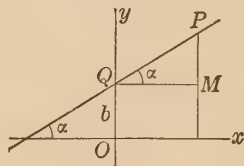


FIG. 27

Formula (1) is known as the *slope form* of the equation of the straight line. It is evidently a special case of the point-slope form—viz. the case $x_1 = 0$, $y_1 = b$.

27. The general equation of first degree. If an equation is of the first degree in x and y , it can contain at most a term in x , a term in y , and a constant term: i.e. it can be written in the form

$$(1) \quad Ax + By + C = 0.$$

If $B = 0$, the equation evidently represents a line parallel to Oy . If $B \neq 0$, the equation can be solved for y : i.e. it can be written in the form

$$(2) \quad y = mx + b.$$

Now for any value of m and any value of b there exists a line with slope m and y -intercept b ; thus it follows from § 26 that equation (2) represents a line. Hence we have proved the

THEOREM: *The locus of every equation of the first degree is a straight line.*

The converse of this theorem is also true: *

The equation of a straight line is always of the first degree.

For, if the line is parallel to Oy , its equation has the form

$$x = k,$$

which is of the first degree. If it intersects the y -axis, it must have a certain slope and a certain y -intercept (either or both may of course be 0), and by § 26 its equation may be written in the form

$$y = mx + b,$$

which is also of the first degree.

* A highly critical reader might take exception to the converse theorem. Take for example the equations

$$x - y = 0, \quad x^2 - 2xy + y^2 = 0, \quad (x - y)(x^2 + y^2 + 1) = 0.$$

These equations all have the same locus, a straight line, yet only one is of the first degree. However, it is natural to take the simplest—i.e. the one of first degree—as “the” equation of the line, and we will agree to do this. A similar remark applies to various theorems later.

28. Reduction to the slope-form. The foregoing theory establishes the

RULE: To reduce the equation of any line to the slope form, solve the equation for y . When this has been done, the coefficient of x is the slope and the constant term is the y -intercept.*

Example: The equation

$$3x + 4y + 6 = 0$$

becomes, in the slope form,

$$y = -\frac{3}{4}x - \frac{3}{2},$$

whence the slope is $-\frac{3}{4}$ and the y -intercept is $-\frac{3}{2}$.

29. Parallel and perpendicular lines. By reduction to the slope form, it is easily seen that the lines

$$Ax + By + C = 0,$$

$$Ax + By + K = 0$$

are parallel, while the lines

$$Ax + By + C = 0,$$

$$Bx - Ay + K = 0$$

are perpendicular. Hence, if a line is to be parallel to a given line, the coefficients of x and y in the required equation may be taken *the same as those in the given equation*; if a line is to be perpendicular to a given line, the coefficients of x and y in the required equation may be found by *interchanging the coefficients of x and y and changing the sign of one of them*. In each case, of course, the constant term must be determined by an additional condition.

Example: Write the equation of a line through the point $(3, -1)$ perpendicular to the line $3x + 2y = 6$.

The left member of the required equation will be

*Excepting, of course, a line parallel to the y -axis. See §§ 8, 23.

$2x - 3y$; if the new equation is to be satisfied by the co-ordinates $(3, -1)$, the right member must be what the left member becomes when those coördinates are substituted. Hence the answer is

$$2x - 3y = 9.$$

EXERCISES

Reduce the following equations to the slope form, and draw the lines.

1. $3x - 2y + 4 = 0.$

2. $x + 3y - 5 = 0.$

3. $4x + 5y + 1 = 0.$

4. $2x - 5y + 3 = 0.$

5. $7x - 4y = 0.$

6. $2x + 3y = 0.$

Write, at sight, the equations of the following lines. (§ 29.)

7. Through $(4, 2)$ (a) parallel, (b) perpendicular to the line $y = 3x$.

8. Through $(3, 1)$ (a) parallel, (b) perpendicular to the line $2x + y = 3$.

9. Through $(-1, 5)$ (a) parallel, (b) perpendicular to the line $2x - 3y = 4$.

10. Through $(4, -1)$ (a) parallel, (b) perpendicular to the line $3x + 4y + 1 = 0$.

11. Having the x -intercept 3 and (a) parallel, (b) perpendicular to the line $2x - 5y = 5$.

✓ 12. Having the x -intercept -1 and (a) parallel, (b) perpendicular to the line $x - 6y = 1$.

13. Through $(1, 3)$ (a) parallel, (b) perpendicular to the line $3x + 5 = 0$.

14. Through $(-1, 2)$ (a) parallel, (b) perpendicular to the line $2y = 5$.

15. Prove that the lines $x + 2y + 1 = 0$, $6x - 3y = 5$, $y = 2x - 1$, and $4x + 8y + 7 = 0$ form a rectangle.

16. Prove that the lines $2x - 3y + 2 = 0$, $4x - 2y = 3$, $4x - 6y = 1$, $y = 2x + 2$ form a parallelogram.

17. Find the locus of centers of circles touching the line $2x + 3y = 5$ at $(4, -1)$.

18. Find the locus of centers of circles touching the line $3x + y = 9$ at $(1, 6)$.

19. Prove that a circle can be drawn touching the lines $x - y + 4 = 0$ and $7x + y = 8$ at $(3, 7)$ and $(1, 1)$ respectively.

20. Can a circle be drawn touching the lines $x + y = 2$ and $7x + 20y = 9$ at $(-1, 3)$ and $(7, -2)$ respectively?

30. Linear functions. If two variables y and x are so related that, *when the value of x is given, the value of y is determined*, then y is said to be a *function of x* .

Countless examples of functional relationship are already familiar to the student. The volume of a sphere is a function of—in other words, is determined by—the radius; the strength of a piece of copper wire depends upon its size; the value of a diamond of given quality depends upon its weight.

When x and y are connected by an equation of the *first degree*, we say that y is a *linear function of x* . Evidently every linear function may be written in the form

$$y = mx + b.$$

For the present, we shall be concerned only with linear functions; examples of other types will occur later.

The variable x is called the *independent variable*, or *argument*. It is evidently possible to reverse the roles of the two variables: by merely solving for x , we may express x as a function of the argument y .

Examples: (a) A car drives from A to B , 200 miles distant, at 40 mi. per hr. The distance from A (in miles) after time t (in hours) is

$$s = 40t.$$

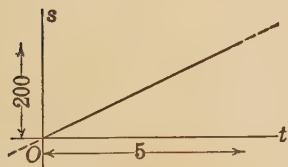


FIG. 28

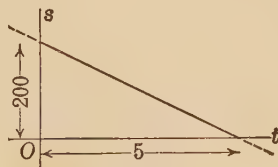


FIG. 29

(b) In (a), the distance from B is

$$s = 200 - 40t.$$

(c) The amount of one dollar at 5% simple interest, after t years, is

$$A = 1 + 0.05t.$$

Any function may be represented graphically by merely plotting the argument as abscissa and the function as ordinate. Evidently the graph of a linear function is a straight line, but with this limitation: Due to the nature of the problem, the graph may have a meaning only in a restricted range, or interval, of values of the argument. Thus in (a), (b)

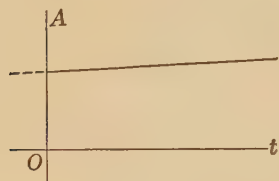


FIG. 30

above, we are concerned only with the interval $0 \leq t \leq 5$; in (c), only with the interval $t \geq 0$. The graphs of these three functions are shown in Figs. 28, 29, 30 respectively.

31. Rate of change. If y is a function of x , it is clear that when x changes from any given value x_1 to a new value x_2 , y will change from its original value y_1 to the new value y_2 . It will be convenient to denote the *change in x* by the symbol Δx (read delta x), the corresponding *change in y* by the symbol Δy : that is,

$$x_2 - x_1 = \Delta x, \quad y_2 - y_1 = \Delta y.$$

This notation is commonly used in more advanced mathematics.

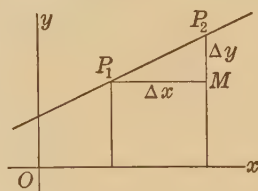


FIG. 31

If y is a *linear* function of x , it is easily seen that when x changes by any amount, *the change in y is proportional to the change in x* . For, as we pass from any point P_1 to any other point P_2 on the graph, the change in x is P_1M , the change in y is MP_2 , and

$$\frac{\Delta y}{\Delta x} = \frac{MP_2}{P_1M} = m,$$

which is constant *—that is, will have the same value no matter what two points P_1, P_2 are chosen.

The ratio of the change in y to the change in x is the *rate of change* of the linear function.† We have at once the

THEOREM: *The rate of change of a linear function is constant, and equal to the slope of its graph.*

Further, a positive rate (slope) means that the function is *increasing*; negative rate, *decreasing*. See for example Figs. 28–30.

EXERCISES

Express the function by a formula and draw the graph, indicating that portion of the graph that has a meaning in each case. Determine the rate of change.

1. The distance covered by a man running 8 yd. per sec., as a function of time.
2. The value of a consignment of grain at 60¢ per bu., as a function of the number of bushels.
3. The function of Ex. 2, if \$100 must be deducted for transportation charges.
4. The value of a farm at \$50 per acre, with buildings worth \$2000.
5. The diagonal of a square as a function of the length of the side.
6. The total area (including both bases) of a right circular cylinder of radius unity, as a function of the altitude.
7. The total area (including the base) of a right circular cone of radius unity, as a function of the slant height.
8. The length of the diagonal of a cube as a function of the length of the edge.
9. The radius of a circle increases 1 ft. How much does the circumference increase if the original radius is (a) 1 inch? (b) 1 mile?

* To say that any quantity u is proportional to another quantity v means that $u = kv$, or $\frac{u}{v} = k$, where k is constant. Here $u = \Delta y$, $v = \Delta x$, $k = m$.

† It must be firmly fixed in mind, however, that this statement applies to linear functions only.

In Exs. 10–13, assume that B is 200 mi. east of A .

10. A car, starting at noon, drives east from A at 30 mi. per hr. A second car, starting at 2:00 P.M., drives east from A at 50 mi. per hr. Find analytically and graphically (a) when and where the cars will be together; (b) their relative positions when the faster car reaches B .

11. A car, starting at noon, drives east from A at 40 mi. per hr. A second car, starting at 3:00 P.M., drives east from B at 25 mi. per hr. Find analytically and graphically when and where they will be together.

12. A car, starting at noon, drives east from A at 50 mi. per hr. A second car, starting at 1:00, drives west from B at 40 mi. per hr. Find analytically and graphically when and where they will meet.

13. A car, starting at noon, drives east from A at 40 mi. per hr. A second car, starting at 1:00, drives west from B at 30 mi. per hr. A third car, starting at 2:00, drives west from B at 50 mi. per hr. Determine analytically and graphically the period during which the first car will be between the other two. Ans. 3:17–3:20.

32. The intercept form. Let the intercepts of a straight line on the axes be $OA = a$ and $OB = b$. Choosing a point $P : (x, y)$ in a general position on the line, we see that the triangles MAP and OAB are similar. Whence

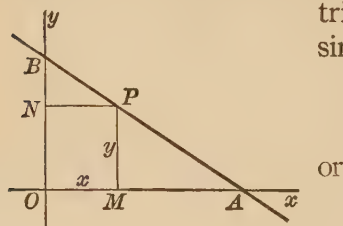


FIG. 32

$$\frac{MP}{OB} = \frac{MA}{OA},$$

$$\frac{y}{b} = \frac{a - x}{a}.$$

The equation is usually written in the equivalent form

$$(1) \quad \frac{x}{a} + \frac{y}{b} = 1,$$

called the *intercept form*.

To write the equation of any line in the intercept form we have merely to find the intercepts and substitute in (1). This form evidently fails in case the line passes through the origin or is parallel to either axis.

Example: Reduce the equation

$$3x - 2y + 8 = 0$$

to the intercept form.

Setting $y = 0$, we find the x -intercept to be $-\frac{8}{3}$; the y -intercept is 4. Hence the intercept form is

$$\frac{x}{-\frac{8}{3}} + \frac{y}{4} = 1.$$

EXERCISES

Write the equations of the following lines, and draw the lines.

1. Intercepts 4, -1 .

2. Intercepts -2 , -3 .

3. Intercepts $\frac{3}{2}$, $-\frac{1}{2}$.

4. Intercepts $\frac{4}{3}$, $\frac{4}{3}$.

Reduce the following equations to the intercept form. Draw the lines.

5. $3x + 2y = 6$.

6. $5x + 2y = 20$.

7. $2x + 3y + 5 = 0$.

8. $4x - 5y + 7 = 0$.

9. $7x - 3y + 8 = 0$.

10. $3x + 9y + 1 = 0$.

11. A line makes equal intercepts on the axes and passes through $(7, 3)$. Find its equation in two ways. (§§ 32, 24.)

12. A line makes equal intercepts on the axes and passes through $(-2, 1)$. Find its equation in two ways.

13. The intercepts are numerically equal but of opposite sign; the line passes through $(-3, -4)$. Find its equation by two methods.

14. The y -intercept is twice the x -intercept; the line passes through $(4, -3)$. Find its equation.

15. The product of the intercepts is 1; the line passes through $(6, -1)$. Find its equation. *Ans.* $x + 4y = 2$; $x + 9y + 3 = 0$.

16. A line passes through $(-4, 1)$ and forms with the axes a triangle of area 1. Find its equation. *Ans.* $x + 2y + 2 = 0$; $x + 8y = 4$.

17. A line passes through $(-7, 2)$, and the segment of the line intercepted between the axes is of length $5\sqrt{2}$. Find its equation. *Ans.* $x + 7y = 7$; $x + y + 5 = 0$.

18. A rectangle is inscribed in a right triangle of base b and height h . What relation must hold between the base and altitude of the rectangle?

19. A right circular cylinder is inscribed in a right circular cone of radius r and height h . What relation must hold between the radius and height of the cylinder?

20. Derive the intercept form from the fact that, in Fig. 32, the area of the rectangle plus the area of the small triangles equals the area of the large triangle.

21. Derive the intercept form from the fact that the area of the triangle APB is 0. (§ 10.)

33. **The normal form.** A straight line is determined if we know the length p of the perpendicular from the origin to the line, together with the angle β which this perpendicular makes with Ox . To find the equation of a line determined in this way, we assume a point $P : (x, y)$ on the line (the line KPL in Fig. 33) and draw auxiliary lines as shown. Now

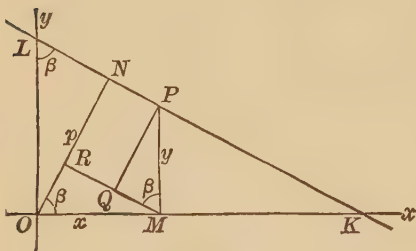


FIG. 33

$$OR + RN = OR + QP \\ = ON.$$

But

$$OR = x \cos \beta, \\ QP = y \sin \beta, \\ ON = p,$$

so that *

$$(1) \quad x \cos \beta + y \sin \beta = p.$$

This is the *normal form* of the equation of the straight line. It is chiefly useful in problems involving the distance of a point from a line, the distance between parallel lines, etc.

For definiteness we shall adopt the conventions that p is *always positive* and β is a *positive angle less than 360°* .

* This method of derivation fails in case the given line passes through the origin. But in that case $p = 0$, and the equation of a line through $(0, 0)$ parallel to (1) is obviously

$$x \cos \beta + y \sin \beta = 0,$$

so that (1) holds in all cases.

To reduce the equation of any line

$$(2) \quad Ax + By + C = 0$$

to the normal form (1), we note that if equations (2) and (1) are to represent the same straight line, the coefficients A, B, C in (2) must be proportional, respectively, to the coefficients $\cos \beta, \sin \beta, -p$ in (1). Let the constant ratio between these numbers be denoted by k :

$$\frac{\cos \beta}{A} = k, \quad \frac{\sin \beta}{B} = k, \quad \frac{-p}{C} = k,$$

so that

$$\cos \beta = kA, \quad \sin \beta = kB, \quad p = -kC.$$

Square and add the first two of these equations:

$$\cos^2 \beta + \sin^2 \beta = k^2(A^2 + B^2),$$

whence

$$k^2(A^2 + B^2) = 1, \quad \text{and} \quad k = \pm \frac{1}{\sqrt{A^2 + B^2}}.$$

Thus

$$(3) \quad \left\{ \begin{array}{l} \cos \beta = \pm \frac{A}{\sqrt{A^2 + B^2}}, \quad \sin \beta = \pm \frac{B}{\sqrt{A^2 + B^2}}, \\ p = \mp \frac{C}{\sqrt{A^2 + B^2}}, \end{array} \right.$$

where the ambiguous sign is to be so chosen that p shall be positive. This leads to the following

RULE: *To reduce the equation of any line to the normal form, divide through by the square root of the sum of the squares of the coefficients of x and y , and choose signs so that the constant term is positive in the right member.**

* When $p = 0$, it is evident that β may have either of two values. This checks with the fact that when $p = 0$, the rule just given fails to determine the ambiguous sign.

Example: Find the locus of points at the distance $\sqrt{5}$ from the line $x - 2y + 10 = 0$.

By the rule, the given equation is, in the normal form,

$$-\frac{x}{\sqrt{5}} + \frac{2y}{\sqrt{5}} = \frac{10}{\sqrt{5}},$$

so that the distance of this line from the origin is $p = \frac{10}{\sqrt{5}}$.

The required locus consists of the two lines parallel to, and at a distance $\sqrt{5}$ from, the given line; the distance of these lines from $(0, 0)$ is $\frac{10}{\sqrt{5}} \pm \sqrt{5}$. Hence the required lines are

$$-\frac{x}{\sqrt{5}} + \frac{2y}{\sqrt{5}} = \frac{10}{\sqrt{5}} \pm \sqrt{5},$$

or

$$-x + 2y = 10 \pm 5.$$

EXERCISES

Reduce the following equations to the normal form, and find the distance of each line from the origin.

1. $4x + 3y - 7 = 0$.

2. $3x - y - 5 = 0$.

3. $x - 4y + 6 = 0$.

4. $5x + 2y + 5 = 0$.

5. $y = 4 - 5x$.

6. $y = 2x - 3$.

7. $5x - 4y = 0$.

8. $x + 3y = 0$.

9. $y = mx + b$.

10. $y - y_1 = m(x - x_1)$.

Find the equations of the following lines.

11. Parallel to $3x + 4y + 25 = 0$ and passing (a) at a distance 2 from the origin; (b) 3 units farther from the origin; (c) at a distance 1 from the given line. *Ans.* (b) $3x + 4y = \pm 40$; (c) $3x + 4y + 25 = \pm 5$.

12. Parallel to $x - y = 6$ and passing (a) at a distance 6 from the origin; (b) $2\sqrt{2}$ units farther from the origin; (c) at a distance $\sqrt{2}$ from the given line. *Ans.* (b) $x - y = \pm 10$; (c) $x - y = 6 \pm 2$.

13. Parallel to $x + 3y + 4 = 0$ and passing at a distance $\sqrt{10}$ from $(-1, 2)$. *Ans.* $x + 3y = 5 \pm 10$.

14. Parallel to $2x - 3y = 6$ and passing at a distance $2\sqrt{13}$ from $(3, 1)$. *Ans.* $2x - 3y = 3 \pm 26$.

15. Parallel to $x - y + 2 = 0$ and passing at a distance 3 from (2, 3).
16. Parallel to $4y = 3x$ and passing at a distance 2 from (3, 1).
17. A circle of radius 3 touches the line $2x - 5y = 2$. Find the locus of its center.
18. A circle of radius 2 touches the line $5x + 12y + 26 = 0$. Find the locus of its center.
19. One side of a square is the line from (0, 5) to (3, 1). Find the other vertices.
Ans. (4, 8), (7, 4); (-4, 2), (-1, -2).
20. The base of a triangle is the line from (6, 4) to (5, 2); the area is $\frac{5}{2}$. Find the locus of the third vertex.
Ans. $2x - y = 13$; $2x - y = 3$.
21. Solve Ex. 20 by means of § 10.
22. In Ex. 20, if the triangle is isosceles, find the third vertex.
Ans. $(\frac{15}{2}, 2)$; $(\frac{7}{2}, 4)$.
23. The base of a triangle joins the points (-2, 4), (-1, 3); the area is $\frac{3}{2}$. Find the locus of the third vertex.
24. Solve Ex. 23 by another method.
25. In Ex. 23, if the triangle is isosceles, find the third vertex.
Ans. (-6, -1); (3, 8).

Find the distance between the given lines.

26. $3x - y = 6$, $3x - y = 8$. *Ans.* $\frac{1}{5}\sqrt{10}$.
27. $x + 2y + 5 = 0$, $x + 2y - 3 = 0$. *Ans.* $\frac{8}{5}\sqrt{5}$.
28. $x + 4y - 1 = 0$, $x + 4y + 6 = 0$.
29. $2x - 3y + 2 = 0$, $2x - 3y + 3 = 0$.
30. Two sides of a square lie in the lines $5y = 3x + 4$, $3x - 5y = 5$. Find the area of the square. *Ans.* $\frac{9}{4}$.
31. Find the area of the rectangle bounded by the lines $x - 2y = 3$, $2x + y + 2 = 0$, $2x + y + 4 = 0$, $x - 2y + 4 = 0$.
32. A circle touches the lines $4x + y + 3 = 0$, $4x + y - 7 = 0$. Find its area, and the locus of its center.

33. In Fig. 33, what are the coördinates of K ? Of L ? Solve by trigonometry directly from the figure, without using the equation of the line.

Ans. $K : (p \sec \beta, 0)$; $L : (0, p \csc \beta)$.

34. Using the result of Ex. 33, derive the normal form from the fact that the area of the triangle KPL is 0.

35. In Fig. 33, what are the coördinates of N ?

36. Derive the normal form from the fact that $\overline{ON}^2 + \overline{NP}^2 = \overline{OP}^2$.

37. Derive the normal form from the fact that ON is perpendicular to NP . (§ 9.)

34. Distance of a point from a line. To find the distance from the line

$$Ax + By + C = 0$$

(the line λ in Fig. 34) to any point $P : (x_1, y_1)$ not on that line, let us suppose the equation reduced to the normal form:

$$x \cos \beta + y \sin \beta = p.$$

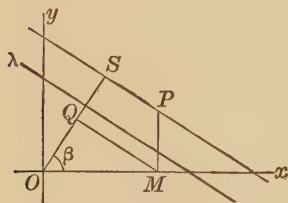


FIG. 34

Then, drawing auxiliary lines as shown, we see that

$$OQ = OM \cos \beta = x_1 \cos \beta,$$

$$QS = MP \sin \beta = y_1 \sin \beta,$$

$$OS = x_1 \cos \beta + y_1 \sin \beta,$$

whence

$$d = x_1 \cos \beta + y_1 \sin \beta - p.$$

Upon substituting in this formula the values of $\cos \beta$, $\sin \beta$, given by (3), § 33, we obtain the following:

The distance from the line

$$Ax + By + C = 0$$

to the point (x_1, y_1) is given by the formula

$$(1) \quad d = \frac{Ax_1 + By_1 + C}{\pm \sqrt{A^2 + B^2}},$$

where the ambiguous sign is to be chosen opposite to the sign of C .

It is clear that the distance as given by the formula will be positive if the point and the origin are on opposite sides of the line, negative if they are on the same side. Usually, however, we are concerned chiefly with the numerical value apart from sign.

Examples: (a) Find the distance from the straight line $3x - 2y + 5 = 0$ to the point $(3, 4)$.

The formula gives at once

$$d = \frac{3 \cdot 3 - 2 \cdot 4 + 5}{-\sqrt{13}} = -\frac{6}{\sqrt{13}}.$$

(b) Find the equations of the lines bisecting the angles between the lines

$$x + y = 2, \quad x - 7y + 2 = 0$$

(the lines λ_1 , λ_2 of Fig. 35).

The bisector of the angle between two lines is the locus of points *equidistant from the two lines*. Assume a point $P : (x, y)$ in the bisector of

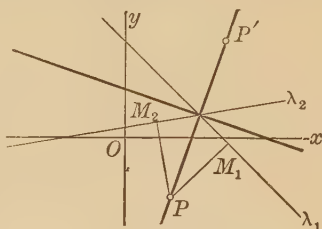


FIG. 35

that angle in which the origin lies. Then the distances M_1P and M_2P are numerically equal; they are also algebraically equal, since in both cases O and P are on the same side of the line and both distances are negative, or when P is in the position P' both are positive. But, by (1),

$$M_1P = \frac{x + y - 2}{\sqrt{2}}, \quad M_2P = \frac{-x + 7y - 2}{5\sqrt{2}},$$

so that the equation of the locus of P is

$$\frac{x + y - 2}{\sqrt{2}} = \frac{-x + 7y - 2}{5\sqrt{2}},$$

or

$$3x - y = 4.$$

In a similar way the equation of the other angle bisector is found to be

$$\frac{x + y - 2}{\sqrt{2}} = -\left(\frac{-x + 7y - 2}{5\sqrt{2}}\right),$$

or

$$x + 3y = 3.$$

EXERCISES

In the following cases, find the distance of the point from the line.

1. $(5, 2)$, $3x - y + 6 = 0$. *Ans.* $-\frac{19}{10}\sqrt{10}$.

2. $(3, -5)$, $2x + y + 4 = 0$. *Ans.* $-\sqrt{5}$.

3. $(4, 3)$, $3x + 2y = 6$. 4. $(0, 5)$, $2y = x + 6$.

5. $(3, -1)$, $x - 4y = 0$. 6. $(-2, 0)$, $3x + y = 0$.

7. By two methods, find the area of the triangle whose vertices are $(3, 1)$, $(5, 2)$, $(-2, 3)$.

8. By two methods, find the area of the triangle whose vertices are $(-1, 1)$, $(4, 2)$, $(6, -1)$.

Find the bisectors of the angles between the given lines.

9. $13x - 9y = 10$, $x + 3y = 6$.
Ans. $2x - 6y + 5 = 0$; $9x + 3y = 20$.

10. $2x + y + 1 = 0$, $11x + 2y = 2$.
Ans. $21x + 7y + 3 = 0$; $x - 3y = 7$.

11. $x = 2$, $y = \frac{3}{4}x$. 12. $x = 3$, $y = 5$.

13. $3x - y = 2$, $y = 3x + 4$. 14. $y = 2x$, $2x - y = 6$.

15. $y = 3x$, $x + y = 1$. *Ans.* $(3 \pm \sqrt{5})x - (1 \mp \sqrt{5})y = \pm \sqrt{5}$.

16. $4x - 5y = 6$, $x + 7y = 8$.
Ans. $5\sqrt{2}(4x - 5y - 6) = \pm \sqrt{41}(x + 7y - 8)$.

17. A circle touches the lines $2x + y = 5$, $x + 2y = 3$. Find the locus of its center.
Ans. $x - y = 2$, $3x + 3y = 8$.

18. A circle touches the x -axis and the line $5y = 12x$. Find the locus of its center.
Ans. $2x - 3y = 0$; $3x + 2y = 0$.

19. A moving point is equidistant from the origin and the line $x + y = 2$. Find the equation of its locus.

Ans. $x^2 - 2xy + y^2 + 4x + 4y - 4 = 0$.

20. A moving point is twice as far from the origin as from the line $x - y = 1$. Find the equation of its locus.

Ans. $x^2 - 4xy + y^2 - 4x + 4y + 2 = 0$.

21. A point moves so that the ratio of its distances from two fixed lines is constant. Prove analytically that its locus is two straight lines.

22. A point moves so that the square of its distance from the point $(1, 0)$ is numerically equal to its distance from the line $x = 1$. Find the equation of its locus. *Ans.* $x^2 + y^2 - x = 0$, $x^2 + y^2 - 3x + 2 = 0$.

23. A point moves so that the product of its distances from two parallel lines is constant. Prove that its locus consists of two lines parallel to the given lines.

35. Angle between two lines. By the *angle from* a line λ_1 to another line λ_2 we shall understand the *acute angle* through which λ_1 must be rotated to come to coincidence with λ_2 . This angle is considered positive if measured counter-clockwise, negative clockwise. In case we are concerned only with the magnitude of the angle without regard to sign, we may speak merely of the angle *between* the lines.

Let the lines λ_1, λ_2 make the (positive or negative) acute angles α_1, α_2 with Ox , and denote by ϕ the angle from λ_1 to λ_2 . Then we have

$$\alpha_2 = \alpha_1 + \phi,$$

whence

$$\phi = \alpha_2 - \alpha_1,$$

and

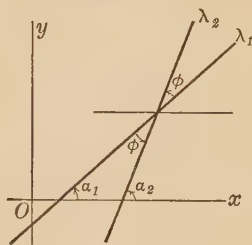


FIG. 36

$$\tan \phi = \tan (\alpha_2 - \alpha_1) = \frac{\tan \alpha_2 - \tan \alpha_1}{1 + \tan \alpha_1 \tan \alpha_2}.$$

But the slopes of the lines are

$$\tan \alpha_1 = m_1, \quad \tan \alpha_2 = m_2,$$

so that the *angle between* * two lines of slopes m_1, m_2 is given by the formula

$$(1) \quad \tan \phi = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

This formula evidently fails if either of the lines is parallel to Oy , but in that case the angle is readily found directly.

Examples: (a) Find the angle between the lines

$$3x - y = 10, \quad 2x + y = 6.$$

* More precisely, the angle from the first line to the second.

The slopes of these lines are *

$$m_1 = 3, \quad m_2 = -2.$$

Substituting in (1), we find

$$\tan \phi = \frac{-2 - 3}{1 - 6} = 1, \quad \phi = 45^\circ.$$

(b) Find the equations of the lines passing through $P : (5, 3)$ and making with the line

$$3x + y = 6$$

(the line λ in Fig. 37) an angle * $\phi = \arctan 2$.

For definiteness, let us measure ϕ from the given to the required line. Then $\tan \phi = 2$, $m_1 = -3$, so that, when ϕ is measured in the positive sense,

$$2 = \frac{m_2 + 3}{1 - 3m_2}.$$

Thus $m_2 = -\frac{1}{7}$, and the desired equation is

$$y - 3 = -\frac{1}{7}(x - 5),$$

or

$$x + 7y = 26.$$

Similarly, measuring ϕ in the negative sense, we write

$$-2 = \frac{m_2 + 3}{1 - 3m_2},$$

so that $m_2 = 1$, and the second required line is

$$y - 3 = x - 5,$$

or

$$x - y = 2.$$

* The symbol $\arctan k$ (also written $\tan^{-1} k$) means an *angle whose tangent is k* : i.e. the statement

$$\phi = \arctan k$$

is equivalent to the statement

$$\tan \phi = k.$$

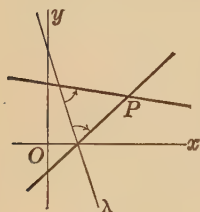


FIG. 37

EXERCISES

Find the angle between the given lines.

1. $y = 3x - 2$, $x + y = 5$. *Ans.* $\arctan 2$.

2. $x + y = 5$, $2x + y = 6$. *Ans.* $\arctan \frac{1}{3}$.

3. $3x + 4y = 0$, $5x - 2y = 3$. *Ans.* $\arctan \frac{26}{7}$.

4. $x - 3y = 2$, $3x + 2y = 4$. *Ans.* $\arctan \frac{11}{3}$.

5. $x + 4y = 3$, $5y = 3x + 2$. *Ans.* $\frac{1}{4}\pi$.

6. $4x - 3y = 6$, $x - 7y = 2$. *Ans.* $\frac{1}{4}\pi$.

7. $3x + 1 = 0$, $2x + 3y = 6$. *Ans.* $\arctan \frac{3}{5}$.

8. $4x = 3$, $5x - 2y = 2$. *Ans.* $\arctan \frac{3}{2}$.

9. Solve Ex. 1, p. 12, using the formula

$$A = \frac{1}{2} bc \sin \alpha,$$

where α is the angle between the sides b , c .

10. Using the formula of Ex. 9, solve Ex. 2, p. 12.

11. Find the equations of the lines through $(0, 0)$ making with the line $x + y = 0$ the angle $\arctan 2$. *Ans.* $y = 3x$, $3y = x$.

12. Find the equations of the lines through $(0, 0)$ making with the line $3y = 2x$ the angle $\arctan \frac{1}{2}$. *Ans.* $4y = 7x$, $8y = x$.

13. Find the equations of the lines through $(2, 1)$ making an angle of 45° with the line $y = 2x + 1$. *Ans.* $3x + y = 7$, $3y = x + 1$.

14. Find the equations of the lines through $(-1, 3)$ making with the line $x + 3y = 2$ the angle $\arctan \frac{1}{4}$. *Ans.* $x + 13y = 38$, $7x + 11y = 26$.

15. Find the equations of the lines through $(9, -6)$ passing at a distance 1 from the point $(4, -1)$. *Ans.* $3x + 4y = 3$, $4x + 3y = 18$.

16. Find the equations of the lines through $(1, 1)$ passing at a distance $\frac{1}{10}\sqrt{10}$ from $(2, 3)$. *Ans.* $y = 3x - 2$, $9y = 13x - 4$.

17. The base of an isosceles triangle joins the points $(-4, 3)$, $(2, 1)$; the altitude is $2\sqrt{10}$. Find the third vertex by using § 35.

Ans. $(1, 8)$, $(-3, -4)$.

18. Solve Ex. 17 by using the normal form.

19. Solve Ex. 17 by using § 10.

20. One side of an isosceles right triangle is the line from $(-2, 16)$ to $(-10, 10)$, with the right angle at $(-2, 16)$. Find the third vertex by several methods. *Ans.* $(4, 8)$, $(-8, 24)$.

CHAPTER V

THE CIRCLE

36. Definitions; standard forms. A *circle* is the locus of a point that moves at a constant distance from a fixed point. The fixed point is the *center*, and the constant distance is the *radius*. The radius as thus defined is of course merely a number of linear units; the term is also used, as in elementary geometry, to mean a line-segment joining the center and a point of the curve. A *diameter* of a circle may mean either a straight line through the center, or the segment of such a line lying inside the curve.

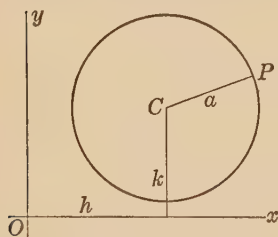


FIG. 38

For the circle with center at O and radius a , we have for any position of the moving point

$$\sqrt{x^2 + y^2} = a;$$

if the center is at any point $C : (h, k)$, as in Fig. 38,

$$\sqrt{(x - h)^2 + (y - k)^2} = a.$$

Hence: *The equation of a circle of radius a is, if the center is the point $(0, 0)$,*

$$(1) \quad x^2 + y^2 = a^2;$$

if the center is the point (h, k) ,

$$(2) \quad (x - h)^2 + (y - k)^2 = a^2.$$

Equations (1) and (2) are called *standard forms* of the equation of the circle. Of course (1) is merely a special case of (2): the case $h = k = 0$.

37. General equation. It appears from (2), § 36, that the equation of a circle is always of the second degree.

The most general equation of the second degree in x and y may contain, at most, terms in x^2 , xy , y^2 , x , y , and a constant: i.e. it may be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Consider now the special case in which $A = C$ and $B = 0$:

$$(1) \quad Ax^2 + Ay^2 + Dx + Ey + F = 0.$$

We may always divide this equation through* by A , transpose the constant term to the right member, and complete the squares in x and y ; the equation then has the form

$$(2) \quad (x - h)^2 + (y - k)^2 = a^2,$$

and consequently represents a circle whenever the right member is positive. Equation (1) is the *general form* of the equation of the circle.

Conversely, it appears from § 36 that the equation of every circle may be put in the form (1).

When an equation of form (1) is reduced to form (2), it may happen that the right member becomes 0:

$$(x - h)^2 + (y - k)^2 = 0.$$

Since this equation holds only when $x = h$ and $y = k$, the locus is the single point (h, k) —a so-called “point-circle.”

Finally, it may happen that the right member of (2) is negative: in this case there is clearly no locus.

Therefore, we have the

THEOREM: *An equation of the second degree in which x^2 and y^2 have equal coefficients and the xy -term is missing represents a circle (exceptionally, a single point, or no locus); and conversely.†*

* If A were 0, we should no longer have a second-degree equation.

† See the footnote, p. 40.

Example: Find the center and radius of the circle

$$4x^2 + 4y^2 - 4x + 2y + 1 = 0.$$

First transpose the constant term to the right member and divide by 4:

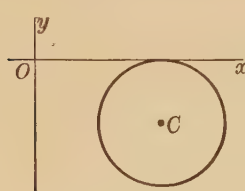


FIG. 39

$$x^2 + y^2 - x + \frac{1}{2}y = -\frac{1}{4}.$$

Then complete the squares in x and y :

$$\begin{aligned} x^2 - x + \frac{1}{4} + y^2 + \frac{1}{2}y + \frac{1}{16} \\ = -\frac{1}{4} + \frac{1}{4} + \frac{1}{16}, \end{aligned}$$

or

$$(x - \frac{1}{2})^2 + (y + \frac{1}{4})^2 = \frac{1}{16}.$$

The center is the point $C : (\frac{1}{2}, -\frac{1}{4})$, and the radius is $\frac{1}{4}$.

EXERCISES

Write the equations of the following circles.

1. With center at $(-5, 3)$, radius $\sqrt{2}$.
2. With center at $(3, -2)$, radius $\sqrt{13}$.
3. With center at $(a, 2a)$, and touching the x -axis.
4. With radius a and touching both axes.
5. With center at $(5, -2)$ and passing through $(4, 3)$.
6. With center at $(-1, -2)$ and passing through $(0, 2)$.
7. With center at $(0, 0)$ and touching the line $2x - y = 5$.
8. With center at $(2, 3)$ and touching the line $x - 3y + 4 = 0$.
9. With center at $(-1, 2)$ and tangent to the line $4x - y = 5$.
10. Using equation (2), § 36, find the condition that a circle shall
 - (a) touch the y -axis;
 - (b) touch both axes;
 - (c) pass through $(0, 0)$;
 - (d) have its center on Ox ;
 - (e) have its center on the line $2x - 3y = 6$.

Draw the following circles.

11. $x^2 + y^2 - 6x + 10y - 2 = 0$.
12. $x^2 + y^2 + 2x + 8y + 8 = 0$.
13. $x^2 + y^2 = 2ax$.
14. $x^2 + y^2 = 3y$.
15. $3x^2 + 3y^2 + 2x - 4y = 0$.
16. $2x^2 + 2y^2 - 3x - 5y + 3 = 0$.

17. Prove that equation (1), § 37, represents a point-circle if and only if $D^2 + E^2 - 4AF = 0$.

18. Prove that (1), § 37, has no locus if and only if $D^2 + E^2 - 4AF < 0$.

19. A point moves so that the sum of the squares of its distances from the points $(a, 0)$, $(-a, 0)$ is constant (equal to k^2). Find the equation of its locus, and draw the curve for the cases $k^2 = 6a^2$, $k^2 = 4a^2$, $k^2 = 3a^2$, $k^2 = 2a^2$, $k^2 = a^2$.

20. Let A, B be two fixed points. A point P moves so that $\overline{AP}^2 + k \cdot \overline{BP}^2 = l^2$, where k, l are constant. What kind of curve is described? Take A, B as $(a, 0)$, $(-a, 0)$. Is the proof general? Discuss special cases.

21. A point moves so that the square of its distance from a fixed point is numerically equal to its distance from a fixed line. What kind of curve is described? Discuss all cases.

22. Taking A, B as in Ex. 20, find the locus of a point P moving so that $AP = k \cdot PB$. Draw the curves $k = \frac{1}{2}$, $k = \frac{9}{16}$, $k = \frac{1}{16}$. What happens as k approaches 0? Approaches 1? Increases indefinitely?

23. Prove analytically that an angle inscribed in a semicircle is a right angle.

24. Prove that the circles $x^2 + y^2 + 4y = 4$, $x^2 + y^2 = 2x + 10y - 8$ are tangent. Draw the figure.

25. Prove that the circles $x^2 + y^2 - 10x - 8y = 4$, $x^2 + y^2 = 2x + 4y$ are tangent. Draw the figure.

26. On the line $x + 2y = 7$, find points at a distance 5 from $(3, 7)$.

Ans. $(3, 2)$, $(-1, 4)$.

27. On the circle $x^2 + y^2 - 4x - 10y + 19 = 0$, find points at a distance 5 from $(5, -1)$.

Ans. $(1, 2)$, $(5, 4)$.

28. Find the points of intersection of the circles $x^2 + y^2 - 2x - 4y - 3 = 0$, $x^2 + y^2 - 4x - 6y - 5 = 0$. What does the result prove?

38. Number of points required to determine a curve.
We know from elementary geometry that a circle is determined by three points. This is proved analytically by the fact that the equation

$$(1) \quad (x - h)^2 + (y - k)^2 = a^2$$

contains three independent constants, h , k , and a . For, when the coördinates of the three points in turn are substituted in (1), we obtain three simultaneous equations to determine h , k , a .

More generally, a circle may be made to satisfy any three conditions which when expressed analytically lead to three simultaneous equations for determining the constants. Thus three tangents may be given, or two points and the radius, etc. It may not, however, be uniquely determined; there may be two or more circles satisfying the given conditions. (For instance, see Exs. 13–16 below.)

The above is merely a special instance of the general

THEOREM: *The number of points required to determine a curve is equal to the number of independent constants in the equation of the curve.*

We have had a previous illustration of this fact in the case of the straight line, which as we know is determined by two points, corresponding to the fact that the linear equation contains two constants.*

39. Circle determined by three conditions. The general method of finding the equation of a circle satisfying three conditions is, as suggested in § 38, to express the conditions analytically by means of three simultaneous equations which may be solved for the constants. The conditions may be expressed most simply in some cases by using the standard form (2) of § 36, in other cases by using the general form (1) of § 37. Very often, however, special methods may be devised which are shorter than the general method and are more easily interpreted geometrically. Such methods can usually be found by recalling the construction of elementary geometry for the center and radius.

* It might seem at first thought that the equation $Ax + By + C = 0$ contains three constants. But if we divide through by any one of them, say A , there remain only two constants, $\frac{B}{A}$ and $\frac{C}{A}$. It is obvious that the essential constants are not the coefficients, but the ratios of any two of them to the third. For example, the equations $y = 2x - 3$, $4x - 2y = 6$, $6y = 12x - 18$ all represent the same line. Similar remarks apply to (1), § 37.

Example: Find the equation of the circle through the points $P_1 : (1, 1)$, $P_2 : (2, -1)$, and $P_3 : (2, 3)$.

One method is to assume the equation of the circle in the form

$$x^2 + y^2 + Dx + Ey + F = 0.$$

(It is convenient to take $A = 1$). Substituting the coordinates of the given points in turn, we get

$$\begin{aligned} 2 + D + E + F &= 0, \\ 5 + 2D - E + F &= 0, \\ 13 + 2D + 3E + F &= 0; \end{aligned}$$

these equations may be solved for D , E , and F , and the results substituted in (1). See also Ex. 25 below.

A second method, which has the advantage of a more direct geometric interpretation, is as follows. The center lies on the perpendicular bisector of P_1P_2 , whose equation is found by familiar methods to be

$$2x - 4y = 3;$$

it also lies on the perpendicular bisector of P_1P_3 , whose equation is

$$2x + 4y = 11.$$

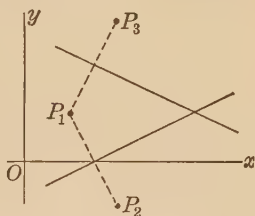


FIG. 40

The center is therefore the point of intersection of these lines, viz. $(\frac{7}{2}, 1)$. The radius is the distance from $(\frac{7}{2}, 1)$ to any one of the given points, which is found to be $\frac{5}{2}$. By formula (2) of § 36, the equation of the circle is

$$(x - \frac{7}{2})^2 + (y - 1)^2 = \frac{25}{4}.$$

Evidently this method merely carries out analytically the elementary geometric construction for the center.

EXERCISES

Find the equations of the following circles. Check the answers.

1. Through $(-1, 2)$, $(3, 4)$, $(2, 3)$. Solve by two methods.
2. Through $(5, 3)$, $(3, 1)$, $(-3, -1)$. Solve by two methods.
3. Circumscribing the triangle $(4, 4)$, $(2, 1)$, $(-1, 3)$.
4. Circumscribing the triangle $(4, 3)$, $(2, 3)$, $(3, 6)$.
5. Passing through the points $(-1, -3)$, $(-5, 3)$, and having its center on the line $x - 2y + 2 = 0$. *Ans.* $(x + 6)^2 + (y + 2)^2 = 26$.
6. Passing through the points $(-1, 1)$, $(-7, 3)$, and having its center on the line $2x + y = 9$. *Ans.* $(x + 1)^2 + (y - 11)^2 = 100$.
7. Touching the line $4x - 3y = 26$ at $(5, -2)$ and passing through $(-2, -3)$. *Ans.* $x^2 + y^2 - 2x - 2y - 23 = 0$.
8. Touching the line $2x + y = 8$ at $(4, 0)$ and passing through $(7, 3)$. *Ans.* $(x - 6)^2 + (y - 1)^2 = 5$.
9. Touching the line $y = 3x$ at $(1, 3)$ and passing through $(5, 7)$.
10. Tangent to $x + y = 1$ at $(4, -3)$ and passing through $(6, 7)$.
11. Touching the x -axis and passing through the points $(4, -1)$, $(-3, -2)$. *Ans.* Centers: $(1, -5)$, $(21, -145)$.
12. Touching the y -axis and passing through the points $(1, 5)$, $(8, 12)$. *Ans.* Centers: $(13, 0)$, $(5, 8)$.
13. Touching the line $x + y = 4$ at $(1, 3)$, and having a radius $\sqrt{2}$. Solve in two ways. *Ans.* Centers: $(0, 2)$, $(2, 4)$.
14. Touching the line $x - 2y = 3$ at $(-1, -2)$, and having a radius $\sqrt{5}$. Solve in two ways. *Ans.* Centers: $(0, -4)$, $(-2, 0)$.
15. Touching the lines $x = 3y$, $3x + y = 2$, and having its center on the line $x + y = 3$. *Ans.* Centers: $(5, -2)$, $(\frac{4}{3}, \frac{5}{3})$.
16. Touching the lines $x + y = 1$, $7x - y = 5$, and having its center on the line $x + y = 4$. *Ans.* Centers: $(-\frac{3}{4}, \frac{19}{4})$, $(3, 1)$.
17. Having a radius $\sqrt{10}$, through $(0, 3)$, $(2, 5)$.
18. Ex. 17 by a method based on § 35.
19. Ex. 17 by a third method.
20. Having a radius $\frac{5}{2}\sqrt{2}$, through $(1, 0)$, $(3, 6)$. Solve in various ways.
21. Inscribed in the triangle formed by the lines $3x - 4y = 5$, $4x - 3y + 10 = 0$, $y = 2$. *Ans.* Center: $(-\frac{5}{21}, \frac{19}{21})$.
22. Touching the lines $x + 2y = 4$, $x + 2y = 2$, $y = 2x - 5$. *Ans.* Centers: $(3, 0)$, $(\frac{1}{5}, \frac{6}{5})$.
23. Touching the circle $x^2 + y^2 = 100$ at $(6, -8)$ and having a radius 15. *Ans.* Centers: $(-3, 4)$, $(15, -20)$.

24. Touching the circle $x^2 + y^2 + 2x - 6y + 5 = 0$ at $(1, 2)$ and passing through $(4, -1)$. Ans. Center: $(3, 1)$.

25. Prove that the equation

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

represents the circle through (x_1, y_1) , (x_2, y_2) , (x_3, y_3) .

26. Solve Exs. 1-4 by the formula of Ex. 25.

27. When does the equation of Ex. 25 reduce to an equation of first degree? Interpret geometrically.

40. Tangents to a given circle.*

I. Two tangents may be drawn to a circle in any given direction—i.e. with any given slope. To find the equations of tangents determined in this way, we proceed as in

Example (a): Find the equations of tangents to the circle

$$x^2 + y^2 - 2x + 4y = 0$$

parallel to the line

$$3x - y = 2.$$

The equation of the circle in the standard form is

$$(x - 1)^2 + (y + 2)^2 = 5,$$

so that the center is $(1, -2)$, the radius is $\sqrt{5}$. The equation of a line through the center and parallel to the given line is evidently

$$3x - y = 5.$$

The required tangents are parallel to this line at a distance $\sqrt{5}$ (the radius) from it: hence, by § 33, their equations are

$$\frac{3x - y}{\sqrt{10}} = \frac{5}{\sqrt{10}} \pm \sqrt{5},$$

* While not necessary, it is advisable that the student read § 75 at this point.

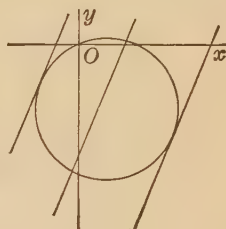


FIG. 41

or

$$3x - y = 5 \pm 5\sqrt{2}.$$

Another general method applicable to this problem will be presented in § 81.

II. To find the tangent at a given point of contact, we merely make use of the fact that *the tangent is perpendicular to the radius drawn to the point of contact*.

III. To find the tangents through a given external point, various methods based on known properties of the circle readily come to mind. One method is shown in

Example (b): Find the tangents to the circle

$$x^2 + y^2 = 20$$

through the point $P: (6, 2)$.

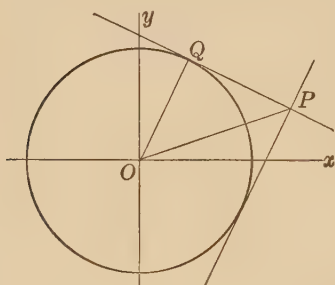


FIG. 42

Write the equation (§ 24) of an arbitrary line through P :

$$y - 2 = m(x - 6),$$

or

$$mx - y = 6m - 2.$$

This line will be tangent to the circle if it passes at a distance $2\sqrt{5}$ (the radius) from the center. Therefore, find the distance of the line from the center (§ 34), and equate this distance to the radius:

$$\frac{6m - 2}{\sqrt{m^2 + 1}} = 2\sqrt{5}, \quad \text{or} \quad \begin{cases} 2m^2 - 3m - 2 = 0, \\ \vdots \end{cases}$$

whence $m = 2$ or $-\frac{1}{2}$, and the tangents are

$$2x - y = 10, \quad x + 2y = 10.$$

Additional methods are suggested in Exs. 25–28 below.

EXERCISES

1. Find the tangents to the circle $x^2 + y^2 = 17$ parallel to the line $4x - y = 5$.
Ans. $4x - y = \pm 17$.

2. Find the tangents to the circle $x^2 + y^2 = 29$ parallel to the line $2x + 5y = 2$.
Ans. $2x + 5y = \pm 29$.

3. Find the tangents to the circle $2x^2 + 2y^2 = 1$ perpendicular to the line $x + 2y = 5$.
Ans. $2y = 4x \pm \sqrt{10}$.

4. Find the tangents to the circle $3x^2 + 3y^2 = 4$ perpendicular to the line $2x - 3y = 2$.
Ans. $9x + 6y = \pm 2\sqrt{39}$.

5. Find the tangents to the circle $4x^2 + 4y^2 - x - y = 1$ parallel to the line $3x - 2y = 4$.
Ans. $24x - 16y = 1 \pm 3\sqrt{26}$.

6. Find tangents to the circle $x^2 + y^2 + 10x - 6y = 2$ parallel to the line $2x - y = 0$.
Ans. $2x - y + 13 = \pm 6\sqrt{5}$.

7. Find tangents to the circle $x^2 + y^2 = 2y$ perpendicular to the line $2x - y = 3$.
Ans. $x + 2y = 2 \pm \sqrt{5}$.

8. Find tangents to the circle $2x^2 + 2y^2 = y$ perpendicular to the line $3x + 3y = 4$.
Ans. $4y = 4x + 1 \pm \sqrt{2}$.

9. Prove in two ways that the line $x + y = 9$ is tangent to the circle $x^2 + y^2 - 4x - 6y + 5 = 0$.

10. Prove in two ways that the line $2y = x + 10$ is tangent to the circle $x^2 + y^2 + 6x - 2y + 5 = 0$.

11. Find the equation of a circle with center at $(1, 7)$ tangent to the circle $x^2 + y^2 = 2$.
Ans. Radii: $4\sqrt{2}, 6\sqrt{2}$.

12. Find the equation of a circle with center at $(1, -2)$ touching the circle $x^2 + y^2 = 45$.
Ans. Radii: $2\sqrt{5}, 4\sqrt{5}$.

13. Prove that the lines

$$y = mx \pm a \sqrt{1 + m^2}$$

are tangent to the circle $x^2 + y^2 = a^2$ for all values of m .

14. Use the formula of Ex. 13 to verify the answers to Exs. 3, 4.

Find the tangent to the circle at the given point on the circle.

15. $x^2 + y^2 = 26$ at $(-1, -5)$. *Ans.* $x + 5y + 26 = 0$.

16. $x^2 + y^2 = 34$ at $(-3, 5)$. *Ans.* $3x - 5y + 34 = 0$.

17. $2x^2 + 2y^2 = 5x$ at $(2, -1)$. *Ans.* $3x - 4y = 10$.

18. $x^2 + y^2 = 5y$ at $(-2, 1)$. *Ans.* $4x + 3y + 5 = 0$.

19. $x^2 + y^2 - 5x + 2y + 3 = 0$ at $(2, -3)$.

20. $x^2 + y^2 + 2x - y - 17 = 0$ at $(3, -1)$.

21. $4x^2 + 4y^2 + 3x - 9y = 0$ at $(0, 0)$. *Ans.* $x = 3y$.

22. $3x^2 + 3y^2 + 2x + 5y = 0$ at $(0, 0)$. *Ans.* $2x + 5y = 0$.

23. Given a point (x_1, y_1) on the circle $x^2 + y^2 = a^2$, prove that the tangent at (x_1, y_1) is

$$x_1x + y_1y = a^2.$$

24. Use the formula of Ex. 23 to verify the answers to Exs. 15, 16.

25. Solve example (b), § 40, by carrying out analytically the familiar construction of elementary geometry: Find M , the midpoint of OP ; draw (i.e. write the equation of) a circle with center at M and radius OM ; find the intersections of this with the given circle.

26. In Fig. 42, the distances OP , OQ are known, so that the distance PQ can be found. Solve example (b) by striking a circle with center P and radius PQ .

27. In Fig. 42, the tangent of $\angle OPQ$ can be found. Solve example (b) by using § 35.

28. In Fig. 42, the area of the triangle OPQ can be found. Solve example (b) by using § 10.

Find the tangents to the circle through the given point, using various methods.

29. $x^2 + y^2 = 10$ through $(5, 5)$. *Ans.* $3y = x + 10$, $3x - y = 10$.

30. $x^2 + y^2 = 2$ through $(6, 4)$. *Ans.* $x - y = 2$, $17y = 7x + 26$.

31. $x^2 + y^2 = 25$ through $(2, -11)$.

32. $x^2 + y^2 = 50$ through $(10, -20)$.

33. $x^2 + y^2 - 4x + 4y - 2 = 0$ through $(4, 2)$.

Ans. $x - 3y + 2 = 0$, $3x + y = 14$.

34. $x^2 + y^2 + 2x - 6y = 0$ through $(7, -11)$.

Ans. $3x + y = 10$, $31x + 27y + 80 = 0$.

35. $x^2 + y^2 - 4x - 16 = 0$ through $(3, 8)$.

Ans. $2x + y = 14$, $19y = 22x + 86$.

36. $3x^2 + 3y^2 + 5x - y + 2 = 0$ through $(0, 0)$.

Ans. $(5 \pm 4\sqrt{3})x + 23y = 0$.

37. In finding the tangents to a circle by the method of example (b), § 40, what conclusion can be drawn if the values of m turn out to be imaginary? Equal?

38. If we were to employ the method of Ex. 25 in a problem without drawing a figure, and it happened that the given point lay inside the circle, where and how in our analysis would that fact come to light?

39. Answer the question of Ex. 38 as regards the method of Ex. 26.

40. Given two points P_1, P_2 on a circle, prove analytically that the distance from P_1 to the tangent at P_2 equals the distance from P_2 to the tangent at P_1 . (Use the formula of Ex. 23).

41. Curves through the intersections of two given curves. Let there be given any two curves, whose equations we will denote by

$$(1) \qquad u = 0, \qquad v = 0,$$

where u and v represent certain expressions* involving x and y . Suppose these curves intersect in certain points P_1, P_2 , etc.

We now take for consideration the locus

$$(2) \qquad u + kv = 0,$$

where k is an arbitrary constant, and proceed to show that, for all values of k , this curve passes through the points of intersection of the two given curves.

Since P_1 lies on the curve $u = 0$, its coördinates when substituted for x and y in the expression u , no matter where u is found, will make that quantity identically zero. Similarly, since P_1 lies on the curve $v = 0$, its coördinates will make v vanish. Therefore, substituting the coördinates of P_1 in (2), we get

$$0 + k \cdot 0 = 0,$$

which is true for all values of k ; hence P_1 lies on the curve (2). By the same reasoning this curve passes through all the other points of intersection of the given curves.

In summary, we have proved the

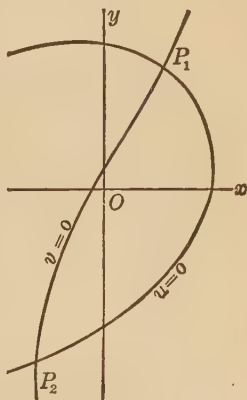


FIG. 43

*The student should make sure that he understands the meaning of the abbreviated notation here used. The letters u and v stand for *any expressions* in x and y whatsoever, not necessarily of the first or second degree.

THEOREM: *If $u = 0$ and $v = 0$ are two intersecting curves, the equation*

$$(3) \quad u + kv = 0,$$

where k is an arbitrary constant, represents for any value of k a curve passing through the points of intersection of the given curves.

Since equation (3) contains one undetermined constant, the curve may be made to satisfy one condition (§ 38)—for instance to pass through a given point or to touch a given line.

When $u = 0$, $v = 0$ are two circles—i.e. when u and v stand for two expressions of the form $Ax^2 + Ay^2 + Dx + Ey + F$ —it is easily seen that in general the equation

$$(4) \quad u + kv = 0$$

is of the form (1), § 37, and hence represents a circle.* Since this circle must pass through the two points of intersection of the given circles, the fact that the new circle is determined by one more condition appears independently of the constant-counting theorem.

Example: Find the circle through the points of intersection of the circles

$$x^2 + y^2 = y, \quad x^2 + y^2 = x$$

and through the point $P : (1, \frac{1}{2})$. (Fig. 44.)

By (4), the required equation may be written in the form

$$(5) \quad x^2 + y^2 - y + k(x^2 + y^2 - x) = 0.$$

If this circle is to pass through $(1, \frac{1}{2})$, those coördinates

* There is one exceptional case; in Ex. 19 below the student is asked to discover this case.

It should also be noted that even when the given circles do not intersect, equation (4) will still be of form (1), § 37 (except as just remarked), and therefore will in general represent a circle.

may be substituted for x and y :

$$1 + \frac{1}{4} - \frac{1}{2} + k(1 + \frac{1}{4} - 1) = 0,$$

whence

$$k = -3.$$

Substituting this value of k in (5), we get

$$x^2 + y^2 - y - 3(x^2 + y^2 - x) = 0,$$

or

$$2x^2 + 2y^2 - 3x + y = 0,$$

which is the desired result.

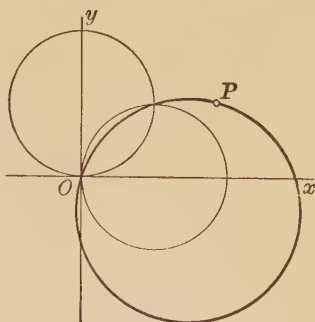


FIG. 44

EXERCISES

Find the equations of the following straight lines.

1. Through the intersection of the lines $3x - y + 2 = 0$, $x + 2y = 3$ and through $(1, 2)$. Check by actually finding the point of intersection of the lines. *Ans.* $8y = 3x + 13$.

2. Through the intersection of the lines $2x + 3y + 1 = 0$, $6x - 5y = 3$ and through $(-2, -2)$. Check by actually finding the point of intersection of the lines. *Ans.* $11x - 15y = 8$.

3. Through the intersection of the lines $3x - 4y = 6$, $y = 2x - 3$ and through $(1, 2)$.

4. Through the intersection of the lines $2x - 3y + 8 = 0$, $y = 3x + 2$ and through $(5, 1)$.

5. Through the intersection of the lines of Ex. 4, and parallel to the y -axis.

6. Through the intersection of the lines of Ex. 4, and parallel to the x -axis.

7. Through the intersection of the lines of Ex. 4, and having the slope 2. *Ans.* $7y = 14x + 16$.

8. Through the intersection of the lines $x - 3y = 6$, $4x + y = 0$, and having the slope -3 . *Ans.* $39x + 13y + 6 = 0$.

9. What is represented by equation (3), § 41, when the original curves are two parallel lines?

10. If $u = 0$ represents a circle and $v = 0$ represents a line, what is the locus of the equation $u + kv = 0$?

Find the equations of the following circles.

11. Through the intersections of the circles $x^2 + y^2 - x + y = 2$, $x^2 + y^2 = 5$, and through $(2, -2)$. *Ans.* $x^2 + y^2 - 3x + 3y + 4 = 0$.

12. Through the intersections of the circles $x^2 + y^2 - 6x + 2y + 1 = 0$, $x^2 + y^2 = 4x$, and through $(1, 3)$.

Ans. $5x^2 + 5y^2 - 8x - 12y - 6 = 0$.

13. Through the intersections of the circle $x^2 + y^2 = 25$ and the line $2x - 3y = 5$, and through the point $(6, 2)$.

14. Through the intersections of the circle $x^2 + y^2 + 2x + 2y = 0$ and the line $y = x - 1$, and through the point $(3, 1)$.

15. Through the intersections of the circles $x^2 + y^2 + 3x - y = 5$, $2x^2 + 2y^2 - 3x + 2y = 4$, and having its center on the y -axis.

16. Through the intersections of the circles $x^2 + y^2 + x - 2y = 6$, $x^2 + y^2 - 2x + y = 10$, and having its center on the x -axis.

17. Through the intersections of the circles $x^2 + y^2 = 2x$, $x^2 + y^2 = 2y$, and having a radius $\sqrt{5}$.

Ans. $x^2 + y^2 - 4x + 2y = 0$; $x^2 + y^2 + 2x - 4y = 0$.

18. Through the intersections of the circles of Ex. 17, and having its center on the line $y = x$.

Ans. $x^2 + y^2 = x + y$.

19. If $u = 0$, $v = 0$ are two non-concentric circles, whether intersecting or not, prove that $u + kv = 0$ is also, in general, a circle. Are there any exceptions?

20. What is represented by $u + kv = 0$ when $u = 0$, $v = 0$ are two concentric circles?

21. If two circles $u = 0$, $v = 0$ are tangent to each other at a point P , prove that $u + kv = 0$ represents, in general, a circle tangent to the given circles at P . Is there any exception?

22. Prove geometrically that, when $u = 0$, $v = 0$ are two non-concentric circles, the center of the circle $u + kv = 0$ is on the line of centers of the given circles. Can the proof be carried out in all cases?

23. Solve Ex. 22 analytically. (For convenience, take the given circles with centers on Ox . Is the proof still general?) What is the advantage of the analytic over the geometric proof?

24. Prove that the curve $u + kv = 0$ passes through no point of either $u = 0$ or $v = 0$ except the points of intersection.

25. In the equation $u + kv = 0$, what happens if we substitute the coördinates of a point on the curve $u = 0$? On the curve $v = 0$?

42. Radical axis. Given any two circles

$$u = 0, \quad v = 0,$$

we may evidently assume without loss of generality that the coefficients of x^2 and y^2 are the same in the two equations. If then we subtract one equation from the other, member by member, the terms of second degree drop out, so that the resulting equation

$$(1) \quad u - v = 0$$

represents a straight line.* This line is called the *radical axis* of the two circles.

In general, the three radical axes of three circles taken in pairs intersect in a point,† called the *radical center* of the three circles. The proof is left to the student.

43. Common chord. If two circles intersect in two distinct points, the line through the points of intersection is called the *common chord*. The term is also used as in elementary geometry, to denote the line-segment joining these points.

It appears at once from § 42 that the radical axis of two intersecting circles is identical with their common chord. For, equation (1) of § 42 is merely that case of (3), § 41, in which $k = -1$. In the special case when the two points of intersection are coincident—i.e. when the circles are tangent to each other—the radical axis is the line tangent to both circles at their point of tangency.

EXERCISES

1. Find the common chord of the circles $x^2 + y^2 - 6x - 8y = 0$, $x^2 + y^2 = 9$. Find the length of the chord. *Ans.* $6x + 8y = 9$; $\frac{2}{3}\sqrt{91}$.

2. Find the common chord of the circles $x^2 + y^2 + 4x - 2y = 0$, $2x^2 + 2y^2 = x$. Find the length of the chord. *Ans.* $4y = 9x$; $\frac{2}{5}\sqrt{97}$.

* There is one exceptional case—see Ex. 9 below.

† Again there is an exception—see Ex. 13 below.

3. Find the common chord of the circles $(x + 2)^2 + (y - 3)^2 = 25$, $(x - 1)^2 + (y + 4)^2 = 9$.

4. Find the common chord of the circles $(x - 3)^2 + (y - 1)^2 = 10$, $3x^2 + 3y^2 = x + y$.

5. Find the radical center of the circles $x^2 + y^2 = 4y$, $x^2 + y^2 = 4$, $x^2 + y^2 - 6x + 8y + 24 = 0$. Draw the figure. *Ans.* (6, 1).

6. Find the radical center of the circles $x^2 + y^2 + 3x - 2y = 4$, $x^2 + y^2 - 2x - y = 6$, $x^2 + y^2 = 1$. Draw the figure. *Ans.* (-1, -3).

7. Find the radical center of the circles $x^2 + y^2 = x + 5y - 1$, $x^2 + y^2 = 2x + 2y$, $x^2 + y^2 + x + 2y = 10$. Draw the figure. *Ans.* (2, 1).

8. Find the radical center of the circles $x^2 + y^2 = 2x + 2y - 2$, $x^2 + y^2 = 0$, $x^2 + y^2 = 4x$. Draw the figure.

9. In what case do two circles have no radical axis?

10. Prove analytically that, in general, the radical axes of three circles taken in pairs meet in a point.

11. Give a geometric construction for the radical axis of two non-intersecting circles, based on the theorem of Ex. 10.

12. Prove analytically that the radical axis of two circles is perpendicular to their line of centers. Hence vary the construction of Ex. 11.

13. In what case do three non-concentric circles have no radical center?

14. Where is the radical axis of two equal circles situated? Prove analytically.

15. Find the radical axis of two point-circles (i.e. circles of radius 0). Solve (a) directly, (b) by reference to Ex. 14.

16. Prove that the perpendicular bisectors of the sides of a triangle meet in a point. (Exs. 10, 15.)

17. Prove the last statement in § 43.

18. Devise a method for finding the equation of the tangent at a point on a circle by considering the given point as a point circle. (Ex. 17.)

By the method of Ex. 18, find the tangent to the circle at the given point.

19. $x^2 + y^2 - 3x + 5y = 12$ at (1, 2).

20. $x^2 + y^2 + 6x - 3y = 31$ at (3, -1).

21. $3x^2 + 3y^2 - 5x + 7y + 4 = 0$ at (1, -2).

22. $2x^2 + 2y^2 + 3x + y = 20$ at (2, -2).

23. $x^2 + y^2 = a^2$ at (x_1, y_1) . (Cf. Ex. 23, p. 68.)

CHAPTER VI

THE CONIC SECTIONS

44. Definitions. The path of a point which moves so that *its distance from a fixed point is in a constant ratio to its distance from a fixed line* is called a *conic section*, or simply a *conic*. (See also § 46.)

The fixed point is called the *focus* of the conic, the fixed line the *directrix*, and the constant ratio the *eccentricity*. In Fig. 45, if F is the focus, λ the directrix, and P a point on the conic, then the definition evidently says that

$$\frac{FP}{LP} = e,$$

or

$$(1) \quad \rho = ed,$$

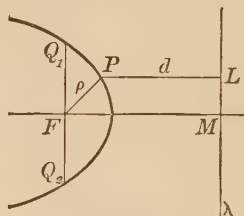


FIG. 45

where e denotes the eccentricity.

It is geometrically obvious that the line through the focus perpendicular to the directrix is an axis of symmetry for the curve. It follows that the line through the focus parallel to the directrix intersects the curve in two points; the chord Q_1Q_2 joining these points is the *latus rectum*.

45. Species of conics. The conic sections fall into three classes, differing greatly in form and in certain of their properties, and distinguished by the value of e , as follows:

if $e < 1$, the conic is an *ellipse*; *

if $e = 1$, the conic is a *parabola*;

if $e > 1$, the conic is a *hyperbola*.*

* For alternative definitions of the ellipse and hyperbola, see §§ 53, 59.

We shall find that the ellipse has the form of a closed oval, the parabola is an open curve extending to infinity, and the hyperbola consists of two separate infinite branches.



FIG. 46

46. The circle; degenerate conics. In addition to the three typical “conics” just discussed, it will be convenient to include under that term various other loci, some of which do not satisfy the definition of § 44.

First, when $e = 0$ the definition fails. But when e approaches 0, the ellipse becomes more and more nearly circular; we will therefore agree to consider the circle as a limiting case of the ellipse.

Second, for reasons both geometric and analytic, we will agree to consider the “point-ellipse” (§ 55), two parallel or coincident lines (§ 50), and two intersecting lines (§ 61) as conic sections of exceptional type. These loci are sometimes called *degenerate conics*.

It can be shown that every plane section of a right circular cone is a curve of the class that we have called “conic sections” (provided the forms just discussed are included); this of course is the reason for the name. The student may amuse himself by discovering, intuitively, how the cutting plane must be passed in order to obtain the various sections.*

* Two parallel lines obviously cannot be cut from a cone. They can, however, be cut from a cylinder, which is the form approached by the cone as the vertex recedes indefinitely.

We will also establish in due course the important

THEOREM: *An equation of the second degree represents a conic section (exceptionally, no locus); and conversely.**

The property embodied in the theorem might be, and in some books is, employed as a definition, instead of the one given in § 44:

A *conic section* is a curve whose equation is of the second degree.

This definition automatically includes not merely the three typical conics, but also the exceptional forms listed above.

EXERCISES

Derive the equations of the following conics.

1. Focus $(0, 0)$, directrix $x = 2$, $e = 1$. *Ans.* $y^2 = -4x + 4$.

2. Focus $(0, 0)$, directrix $y = 3$, $e = \frac{1}{3}$. *Ans.* $9x^2 + 8y^2 + 6y = 9$.

3. Focus $(1, 1)$, directrix $x = 0$, $e = \frac{1}{2}$.
Ans. $3x^2 + 4y^2 - 8x - 8y + 8 = 0$.

4. Focus $(3, 2)$, directrix $x + y = 0$, $e = 1$.
Ans. $x^2 - 2xy + y^2 - 12x - 8y + 26 = 0$.

5. A point moves so that its distance from the point (m, n) bears a constant ratio to its distance from the line $x \cos \beta + y \sin \beta = p$. Find the equation of its path.

Ans. $(x - m)^2 + (y - n)^2 = e^2(x \cos \beta + y \sin \beta - p)^2$.

In Exs. 6-13, use the answer to Ex. 5.

6. Prove that a straight line can intersect a conic in not more than two points (§ 20).

7. Prove that two conics can intersect in not more than four points.

8. How many points are required to determine a conic?

9. How many points are required to determine a parabola?

How many points are required to determine a conic:

10. If the focus is given?

11. If the directrix is given?

12. If the eccentricity is given?

13. If the focus and directrix are given?

* See the footnote, p. 40.

I. THE PARABOLA

47. Definitions. The parabola has been defined in § 45 as *the conic section whose eccentricity is 1*; or in other words:

The parabola is the locus of points which are equidistant from a fixed point and a fixed line.

The line through the focus perpendicular to the directrix is called the *axis* of the curve. The point where the axis intersects the curve, i.e. the point midway between the focus and the directrix, is the *vertex* of the parabola. The distance from the vertex to the focus will be denoted by the letter a .

48. Parabola with vertex as origin: first standard form. Let us take the vertex of a parabola as the origin and the focus at $(a, 0)$. Then the axis of the curve is the x -axis

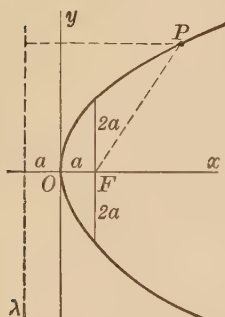


FIG. 47

and the directrix is the line $x = -a$. If $P : (x, y)$ is any point on the curve, then by the definition of the parabola

$$\sqrt{(x - a)^2 + y^2} = a + x,$$

which reduces to

$$(1) \quad y^2 = 4ax.$$

This is the *first standard form* of the equation of the parabola.

From equation (1) it appears that a and x must have the same sign, otherwise their product would be negative and y would be imaginary; hence the curve opens to the right or to the left according as a is positive or negative. For every value of x there are two values of y , numerically equal but of opposite sign, which grow larger as x grows larger: the curve is an open curve extending to infinity.

When $x = a$, $y = \pm 2a$. Hence *the length of the latus rectum is four times the distance between the vertex and focus.*

49. Other standard forms. The equation $y^2 = 4ax$ expresses a characteristic geometric property of the parabola—a property true for all parabolas, regardless of their position with reference to the coördinate axes. Given a parabola with vertex V and focus F (Fig. 48), let us drop from P a perpendicular MP to the axis. Then by (1), § 48,

$$(1) \quad \overline{MP}^2 = 4VF \cdot VM;$$

this is the property just mentioned.

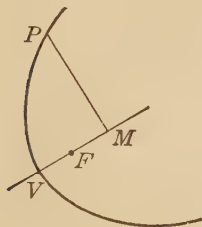


FIG. 48

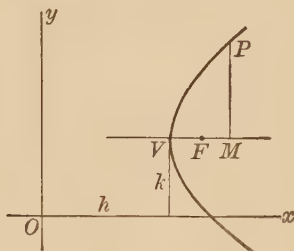


FIG. 49

Consider now a parabola with vertex at (h, k) (Fig. 49), axis parallel to Ox , and focus at a distance a from the vertex. Choosing a point $P : (x, y)$ on the curve, we see at once by (1) that the equation of the parabola is

$$(y - k)^2 = 4a(x - h).$$

We may proceed similarly when the axis is parallel to Oy .

In summary, we have the following result:

The equation of a parabola with vertex at (h, k) is, if the axis is parallel to Ox ,

$$(2) \quad (y - k)^2 = 4a(x - h);$$

if the axis is parallel to Oy ,

$$(3) \quad (x - h)^2 = 4a(y - k).$$

The curve opens in the positive or the negative direction according as a is positive or negative.

EXERCISES

In the following cases, locate the vertex, the ends of the latus rectum, and a few other points, and trace the curve.

- | | |
|---|--|
| 1. $y^2 = -8x$. | 2. $y^2 = 3x$. |
| 3. $x^2 = 6y$. | 4. $x^2 + y = 0$. |
| 5. $4y^2 + x = 0$. | 6. $8x^2 = 3y$. |
| 7. $(x + \frac{1}{2})^2 = \frac{4}{3}(y - 1)$. | 8. $(y - 2)^2 = -12(x + 3)$. |
| 9. $(y + 3)^2 = -2(x + 1)$. | 10. $(x - \frac{1}{4})^2 = \frac{1}{2}y$. |
| 11. $(x - 10)^2 = 100(y - 5)$. | 12. $(y + 6)^2 = 6(x - 8)$. |

Find the equations of the following parabolas.

13. With vertex at O , axis Ox , and passing through $(3, -2)$.
14. With vertex at O , axis Oy , and passing through $(-1, 4)$.
15. With vertex $(3, -1)$ and focus $(3, -2)$.
16. With vertex $(-1, -2)$ and focus $(-4, -2)$.
17. With vertex $(2, 4)$ and directrix $x = 1$.
18. With vertex $(-3, 2)$ and directrix $y + 1 = 0$.
19. With focus $(0, 6)$, axis Oy , and latus rectum 8. (Two answers.)
20. With focus $(2, 3)$, axis parallel to Ox , and latus rectum 12.
21. With vertex on Oy , axis parallel to Ox , and passing through $(-4, 1)$, $(-1, -1)$.
Ans. $(y + 3)^2 = -4x$; $(y + \frac{1}{3})^2 = -\frac{4}{9}x$.
22. With vertex on Ox , axis parallel to Oy , and passing through $(2, 3)$, $(-1, 12)$.
Ans. $(x - 5)^2 = 3y$; $(x - 1)^2 = \frac{1}{3}y$.
23. With vertex on the line $y = 2$, axis parallel to Oy , latus rectum 6, and passing through $(0, 8)$.
Ans. $(x \pm 6)^2 = 6(y - 2)$.
24. With axis parallel to Ox , latus rectum 1, and passing through $(-6, 4)$, $(9, 1)$.
Ans. $(y - 5)^2 = x + 7$; $y^2 = -(x - 10)$.
25. Show in detail how to construct points of a parabola by ruler and compass if the focus and directrix are given.

Solve the following problems geometrically (by ruler and compass).

26. Given a parabola with its vertex, construct the axis.
27. Given a parabola with its focus, construct the directrix.
28. Given the directrix and two points of a parabola, find the focus.
 How many solutions are there, in general? Discuss exceptional cases.
29. Given the focus and two points of a parabola, find the directrix.
30. Given the directrix, the tangent at the vertex, and one point of a parabola, construct the focus.
31. Given a parabola with its vertex, find the focus.

50. General equation. An equation of the form

$$(1) \quad Cy^2 + Dx + Ey + F = 0 \quad (C \neq 0)$$

can evidently be reduced to form (2), § 49, by dividing by C and completing the square in y ; similarly the equation

$$(2) \quad Ax^2 + Dx + Ey + F = 0 \quad (A \neq 0)$$

can be reduced to form (3), § 49. Exception arises when, in (1), $D = 0$ or, in (2), $E = 0$, in which cases the equation represents two parallel or coincident lines,* or has no locus. In summary, we have the

THEOREM: *An equation of the second degree in which the xy -term is missing and only one square term is present represents a parabola with its axis parallel to a coördinate axis (exceptionally, two parallel or coincident lines, or no locus).*

Example: Reduce the equation $4y^2 - 24x - 12y = 15$ to a standard form, and trace the curve.

First divide by 4:

$$y^2 - 6x - 3y = \frac{15}{4}.$$

Transpose the term in x and complete the square in y :

$$y^2 - 3y + \frac{9}{4} = 6x + \frac{15}{4} + \frac{9}{4},$$

or

$$(y - \frac{3}{2})^2 = 6(x + 1).$$

Hence the vertex is at $(-1, \frac{3}{2})$, and the curve opens to the right, with $a = \frac{3}{2}$.

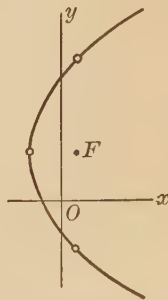


FIG. 50

* Geometrically, of course, two coincident lines cannot be distinguished from a single line. But for various reasons the terminology here adopted is preferable. First, we say in algebra that when $D^2 - 4AF = 0$, the quadratic equation $Ax^2 + Dx + F = 0$ has, not one root, but two equal roots; clearly the analogous statement is that the locus of this equation is not one line but two coincident lines. Second, if we say that the locus is a single line, then the converse theorem of § 27, as stated, is flatly contradicted; and that theorem cannot be stated in any other form without obscuring the essential truth.

EXERCISES

In the following cases, reduce to a standard form, locate the vertex, the ends of the latus rectum, and a few other points, and trace the curve.

1. $y^2 + 8x + 8 = 0$.
2. $y^2 - 5x + 10 = 0$.
3. $x^2 - 4x - 4y = 0$.
4. $x^2 + 6x - y - 2 = 0$.
5. $y^2 + 5x - 3y = 0$.
6. $x^2 - x - y = 3$.
7. $2x^2 + 4x + y + 6 = 0$.
8. $3y^2 - 2x - 6y = 0$.
9. $3x^2 + 2x - y + 1 = 0$.
10. $5x^2 - 6x + 2y + 1 = 0$.
11. $x^2 - 20x + y = 20$.
12. $y^2 - 100x - 200y + 200 = 0$.
13. $12y^2 - 2x - y = 0$.
14. $8x^2 + x + 2y = 1$.
15. $4y^2 - 12y + 9 = 0$.
16. $5x^2 - 6x + 1 = 0$.

17. Find the equation of a parabola with axis parallel to Ox and passing through $(6, 1)$, $(-2, 3)$, $(16, 6)$. *Ans.* $2y^2 - x - 12y + 16 = 0$.

18. Find the equation of a parabola with axis parallel to Ox and passing through $(1, -1)$, $(9, 1)$, $(9, -2)$. *Ans.* $4y^2 - x + 4y + 1 = 0$.

19. Find the equation of a parabola with axis parallel to Oy and passing through $(1, 1)$, $(3, 0)$, $(4, -4)$. *Ans.* $7x^2 - 25x + 6y + 12 = 0$.

20. Find the equation of a parabola with axis parallel to Oy and passing through $(4, 5)$, $(-2, 11)$, $(-4, 21)$. *Ans.* $x^2 - 4x - 2y + 10 = 0$.

21. Find the locus of the center of a circle which passes through $(2, 1)$ and touches the line $x = 10$. *Ans.* $y^2 + 16x - 2y = 95$.

22. Find the equation of a circle through $(0, 5)$, $(3, 4)$, touching the line $y + 5 = 0$. *Ans.* $x^2 + y^2 = 25$; $(x - 60)^2 + (y - 180)^2 = (185)^2$.

23. Find the locus of the center of a circle which touches the circle $x^2 + y^2 = 4$ and the line $x = 3$. *Ans.* $y^2 = 25 - 10x$; $y^2 = 1 - 2x$.

24. A circle touches the line $y = 2$ and the circle $x^2 + y^2 = 16$. Find the locus of its center. *Ans.* $x^2 + 12y = 36$; $x^2 - 4y = 4$.

Trace the given curves, and find their points of intersection.

25. $x^2 + 6x - 3y + 11 = 0$, $x^2 + y^2 + 2x - 2y + 1 = 0$.
26. $y^2 + x - 2y = 0$, $x^2 = y$.
27. $x^2 - 4x - 4y = 4$, $x^2 = 4x + 4y$. *Ans.* No intersection.
28. $x^2 = 3y$, $x^2 = y - 2$.
29. $x^2 + x = y$, $y^2 + 12x = 8y$; *Ans.* $(0, 0)$, $(1, 2)$ twice, $(-4, 12)$.
30. $x^2 - 2x + y + 2 = 0$, $y^2 + 6x + 9y + 2 = 0$.
Ans. $(1, -1)$, $(2, -2)$, $(3, -5)$, $(-2, -10)$.
31. $y^2 - x - 4y + 1 = 0$, $x^2 + 6x - 8y + 25 = 0$.
32. $x^2 - 6x + 2y + 2 = 0$, $y^2 - 2x - 4y + 7 = 0$.

51. Quadratic functions. A function (§ 30) of the form
 (1) $y = ax^2 + bx + c$ ($a \neq 0$)

is called a *quadratic function*. Since this equation is of the form (2), § 50, it appears that the graph of a quadratic function is always a parabola with vertical axis, or (§ 30) a portion of such a parabola.

As x changes, many functions increase to a *maximum* value, decreasing thereafter, or decrease to a *minimum* and then begin to increase. In such cases, it is usually a problem of prime importance to determine the maximum or minimum value. To solve this problem for functions in general, the differential calculus is required; for the quadratic function it may evidently be solved by merely finding the vertex of the parabola (1).

Example: A ball is thrown upward with a velocity of 40 ft. per sec. (a) Find how far it will rise. (b) If it starts at a height of 24 ft., find when it will strike the ground.

When a body moves in a vertical line with no force acting except the earth's attraction, it is shown in mechanics that the distance x (in feet) from the starting point after time t (in seconds) is approximately

$$(2) \quad x = v_0 t - 16t^2,$$

where v_0 is the initial velocity, both x and v_0 being *positive upward*. Take $v_0 = 40$, and standardize (2):

$$(t - \frac{5}{4})^2 = -\frac{1}{16}(x - 25).$$

Thus the ball rises $\frac{5}{4}$ sec., to a height of 25 ft. above the starting point. Putting $x = -24$ in (2), we find

$$2t^2 - 5t - 3 = 0, \quad (2t + 1)(t - 3) = 0,$$

whence the ball strikes the ground after 3 sec.

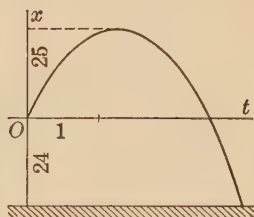


FIG. 51

EXERCISES

In each case draw the curve, and indicate the portion that has a meaning.

1. A ball is thrown upward with a velocity of 64 ft. per sec. Find (a) how far and for how long a time it will rise; (b) when it will be halfway between the starting point and the highest point. *Ans.* (b) 0.6 sec., 3.4 sec.

2. A ball is thrown downward from a height of 10 ft., with a velocity of 12 ft. per sec. Find when it will strike the ground. *Ans.* 0.5 sec.

3. Express the total surface (including both ends) of a right circular cylinder of unit altitude, as a function of the radius.

4. Express the total surface of a right circular cone of slant height 2, as a function of the radius.

5. Find the radius of the circular cross-section cut from a sphere of radius r by a plane passing at a distance h from the center. Taking $r = 1$, express the area of the section as a function of h .

6. Express $\cos 2\theta$ as a function of $\cos \theta$.

7. When the load is uniformly distributed horizontally, a suspension-bridge cable hangs in a parabolic arc. If the bridge is 200 ft. long, the towers 40 ft. high, and the cable 15 ft. above the floor of the bridge at the midpoint, find the equation of the parabola with the midpoint of the bridge as origin; also the height 50 ft. from the middle.

8. A rectangular lot is to be inclosed by 100 yd. of fencing. Express the area as a function of one of the sides, and find the dimensions of the largest lot that can be inclosed. *Ans.* 25×25 yd.

9. A rectangular lot is to be fenced off along the bank of a river. If no fence is needed along the river, find the dimensions of the largest lot that can be inclosed with 100 yd. of fencing. *Ans.* 25×50 yd.

10. A rectangular field is to be inclosed, and divided into three lots by parallels to one of the sides. Find the dimensions of the largest field that can be inclosed with 1000 yd. of fencing. *Ans.* 250×125 yd.

11. A triangular lot has 60 ft. frontage on one street, 80 ft. on another street at right angles to the first. Find the dimensions of the largest rectangular building that can be erected facing one of the streets. *Ans.* 30×40 ft.

12. A right circular cylinder is inscribed in a right circular cone of radius 4 in., altitude 12 in. Find the radius of the cylinder if its convex surface is a maximum. *Ans.* 2 in.

13. Solve Ex. 12 if the total surface of the cylinder is a maximum.

14. A rectangular lot is to be fenced off along a highway. If the fence on the highway costs \$1.50 per yd., on the other sides \$1 per yd., find the dimensions of the largest lot that can be inclosed for \$100. *Ans.* 20×25 yd.

II. THE ELLIPSE

52. Ellipse referred to its axes: first standard form. By § 45, the ellipse is *the conic section for which* $e < 1$.

Let F be the focus and λ the directrix. The line FM through the focus perpendicular to the directrix intersects the curve in two points,* say V, V' . These points are the *vertices*, and C , the midpoint of VV' , is the *center*. Let us set

$$\begin{aligned} CV &= a, \\ CF &= c, \\ CM &= d. \end{aligned}$$

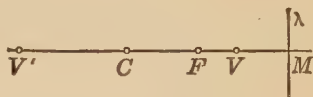


FIG. 52

Then, applying (1), § 44, to the points V, V' , we have

$$a - c = e(d - a), \quad a + c = e(d + a).$$

By subtraction and addition we find

$$\begin{aligned} 2c &= 2CF = 2ae, & CF &= ae, \\ 2ed &= 2e \cdot CM = 2a, & CM &= \frac{a}{e}. \end{aligned}$$

Let us now (Fig. 53) take the center at $(0, 0)$ and focus at $F_1 : (ae, 0)$, so that the directrix is the line

$$x = \frac{a}{e}.$$

Then, if $P : (x, y)$ is a point on the curve, we have (§ 44)

$$\sqrt{(x - ae)^2 + y^2} = e \left(\frac{a}{e} - x \right) = a - ex,$$

$$x^2 - 2aex + a^2e^2 + y^2 = a^2 - 2aex + e^2x^2,$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2),$$

$$(1) \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

For simplicity, let us put

* For FM can be divided both internally and externally in the ratio e .

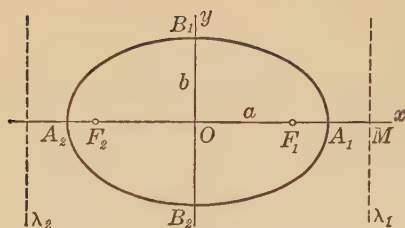


FIG. 53

$$(2) \quad b^2 = a^2(1 - e^2),$$

thus reducing equation (1) to the *standard form*

$$(3) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

From (3) we see that:

(a) The curve is symmetric with respect to Ox and Oy . From the latter statement it follows that there is a *second focus*, the point $(-ae, 0)$, and a *second directrix*, the line $x = -\frac{a}{e}$.

(b) The intercepts on the axes are $(\pm a, 0)$, $(0, \pm b)$.

(c) The equation when solved for y has the form

$$(4) \quad y = \pm \frac{b}{a} \sqrt{a^2 - x^2},$$

which shows that y is imaginary when x is numerically greater than a . Similarly, x is imaginary when y is greater than b .

It is convenient occasionally to speak of the two lines of symmetry as the *axes* of the curve, but usually we shall consider the axes to be the *segments* of these lines included within the curve: the segment A_2A_1 is the *transverse axis*, the segment B_2B_1 the *conjugate axis*.

It follows from (2) that, for the ellipse, b is *always less than* a . Hence the transverse and the conjugate axis are often called the *major* and the *minor* axis respectively.

From equation (3), the ends of the axes may be plotted immediately. By (2), we find

$$ae = \sqrt{a^2 - b^2},$$

so that the distance from center to foci is $\sqrt{a^2 - b^2}$. By (4), when

$$x = ae, \quad y = \pm \frac{b}{a} \sqrt{a^2 - a^2 e^2} = \pm \frac{b^2}{a},$$

so that *the latus rectum* is $\frac{2b^2}{a}$. The ends of the latera recta are then easily plotted; this gives a total of eight points on the curve, from which a fairly accurate sketch can be made.

It is obvious that the equation

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$$

represents the *ellipse with center at the origin and major axis in Oy*.

EXERCISES

In Exs. 1-6, find the center and foci, plot the ends of the axes and of the latera recta, and draw the curve. Find the eccentricity and the directrices.

1. $\frac{x^2}{169} + \frac{y^2}{144} = 1.$

2. $\frac{x^2}{100} + \frac{y^2}{36} = 1.$

3. $\frac{x^2}{2} + \frac{y^2}{3} = 1.$

4. $\frac{x^2}{9} + \frac{y^2}{12} = 1.$

5. $3x^2 + 5y^2 = 15.$

6. $4x^2 + 2y^2 = 9.$

Find the equations of the following ellipses, assuming form (3), § 52.

7. Major axis 9, distance between foci 8.

8. Latus rectum 4, distance between foci $4\sqrt{2}$. *Ans.* $x^2 + 2y^2 = 16.$

9. Eccentricity $\frac{1}{2}$, distance between foci 1. *Ans.* $3x^2 + 4y^2 = 3.$

10. Distance between foci $8\sqrt{6}$, rectangle on the axes of area 80.

11. Eccentricity $\frac{2}{3}$, latus rectum $\frac{8}{3}$. *Ans.* $25x^2 + 45y^2 = 9.$

12. Distance between foci 2, between directrices 8.

Ans. $3x^2 + 4y^2 = 12.$

13. Passing through (4, 3), (6, 2).

Ans. $x^2 + 4y^2 = 52.$

14. Passing through (1, 2), (3, 1).

Ans. $3x^2 + 8y^2 = 35.$

15. Passing through (2, 3), latus rectum three times the distance from center to focus.

Ans. $3x^2 + 4y^2 = 48.$

16. Distance between the directrices $\frac{2}{3}\sqrt{21}$, the rectangle on the axes of area $\frac{8}{7}\sqrt{7}$.

Ans. $4x^2 + 7y^2 = 4; x^2 + 7y^2 = 2.$

17. Latus rectum $\frac{90}{19}$, distance between directrices $2\sqrt{19}$.

Ans. $15x^2 + 19y^2 = 60$; $10x^2 + 19y^2 = 90$.

18. Distance between foci $\frac{4}{3}\sqrt{33}$, passing through $(2, 1)$.

Ans. $x^2 + 12y^2 = 16$.

19. Prove that as e approaches 0, the ellipse approaches the circle as a limiting form. (Let e approach 0 in (1), § 52. As this happens, how do the focus and directrix move?)

20. A line segment of fixed length moves with its ends following two perpendicular lines. The line is divided by a point P into two segments of lengths a, b . Find the locus of P .

Ans. An ellipse.

Solve the following problems by ruler and compass.

21. Given an ellipse with its center, construct the axes.

22. Given an ellipse with its axes, construct the foci.

23. Given an ellipse with its axes, construct the directrices.

53. Another definition of the ellipse. The following property of the ellipse is often used as a *definition*:

An ellipse is the locus of a point which moves so that the sum of its distances from two fixed points is constant. The fixed points are the foci; the constant sum is the transverse axis.

Let the foci be $F_1 : (ae, 0)$, $F_2 : (-ae, 0)$, the directrices

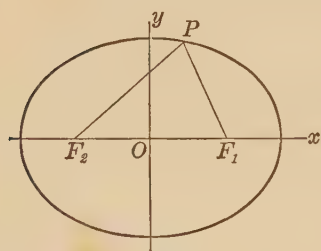


FIG. 54

$x = \pm \frac{a}{e}$, and $P : (x, y)$ any point of an ellipse. By (1), § 44,

$$F_1P = e \left(\frac{a}{e} - x \right) = a - ex,$$

$$F_2P = e \left(\frac{a}{e} + x \right) = a + ex,$$

$$(1) \quad F_1P + F_2P = 2a.$$

To complete the proof of equivalence of the two definitions, it must be shown that, when a point moves so that the sum of its distances from two fixed points is constant, its locus is an ellipse as defined in § 45. This part of the proof may be supplied by the student (cf. Ex. 17, p. 31).

EXERCISES

1. A point moves so that the sum of its distances from $(6, 0)$, $(-6, 0)$ is 16. Find the equation of its locus in two ways (§§ 21, 53). Draw the curve.

2. A point moves so that the sum of its distances from $(4, 0)$, $(-4, 0)$ is 9. Find the equation of its locus in two ways. Draw the curve.

3. A point moves so that the sum of its distances from $(0, 2)$, $(0, -2)$ is 6. Find the equation of its locus in two ways, and draw the curve.

4. A point moves so that the sum of its distances from $(0, 1)$, $(0, -1)$ is 10. Find the equation of its locus in two ways, and draw the curve.

5. A circle touches the circle $(x + 1)^2 + y^2 = 9$, and passes through $(1, 0)$. Find the locus of its center. *Ans.* $20x^2 + 36y^2 = 45$.

6. A moving circle is tangent to the fixed circles $(x - c)^2 + y^2 = c^2$, $(x + c)^2 + y^2 = 16c^2$. Find the locus of its center.

$$\text{Ans. } 84x^2 + 100y^2 = 525c^2; 20x^2 + 36y^2 = 45c^2.$$

7. A moving circle is tangent to the fixed circles $x^2 + (y - 2c)^2 = c^2$, $x^2 + (y + 2c)^2 = 25c^2$. Find the locus of its center. Draw the figure.

$$\text{Ans. } 9x^2 + 5y^2 = 45c^2; x = 0.$$

8. Given the foci and transverse axis of an ellipse, show how to construct points of the curve by ruler and compass.

9. Given the foci and one point of an ellipse, construct the axes.

10. Given the foci and length of the minor axis of an ellipse, construct the major axis.

11. Given one focus, the direction of the major axis, and the lengths of both axes, construct the other focus.

54. Other standard forms. By a method similar to that of § 49, we may easily establish the following:

The equation of an ellipse with center at (h, k) is, if the transverse axis is parallel to Ox ,

$$(1) \quad \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1; \quad (a > b)$$

if the transverse axis is parallel to Oy ,

$$(2) \quad \frac{(y - k)^2}{a^2} + \frac{(x - h)^2}{b^2} = 1. \quad (a > b)$$

(See Figs. 55–56, p. 90.)

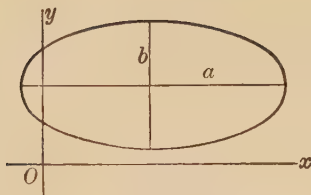


FIG. 55

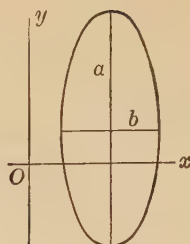


FIG. 56

55. General equation. Given an equation of the form

$$(1) \quad Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

where A and C have *the same sign*, it is easily seen that, by completing the squares in x and y , the equation can in general be reduced to one of the standard forms of § 54. There are two exceptional cases: When the left member is written as the sum of two squares, the right member may be 0, or it may be negative. In the former case the locus is evidently the single point (h, k) ,—the so-called “point-ellipse”; in the latter case there is no locus. Hence the

THEOREM: *An equation of the second degree in which the xy -term is missing and the coefficients of x^2 and y^2 have the same sign represents an ellipse with axes parallel to the coördinate axes (exceptionally, a single point, or no locus).*

Example: Trace the curve

$$9x^2 + 4y^2 - 36x + 8y + 31 = 0.$$

Transpose the constant and complete the squares:

$$9(x^2 - 4x + 4) + 4(y^2 + 2y + 1) = -31 + 36 + 4,$$

$$9(x - 2)^2 + 4(y + 1)^2 = 9,$$

$$\frac{(x - 2)^2}{1} + \frac{(y + 1)^2}{\frac{9}{4}} = 1.$$

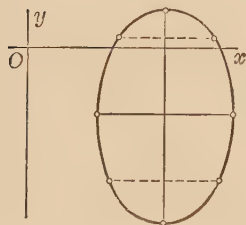


FIG. 57

The center is at $(2, -1)$, transverse axis parallel to Oy , semi-axes $a = \frac{3}{2}$, $b = 1$, distance from center to foci $\sqrt{a^2 - b^2} = \sqrt{\frac{5}{4}} = \frac{1}{2}\sqrt{5}$, latus rectum $\frac{2b^2}{a} = \frac{4}{3}$.

EXERCISES

In each of the following cases, find the center and foci, plot the ends of the axes and of the latera recta, and draw the curve.

1. $\frac{(x-3)^2}{4} + \frac{(y+1)^2}{3} = 1.$

2. $\frac{(x+\frac{3}{4})^2}{\frac{9}{4}} + \frac{(y-1)^2}{\frac{1}{4}} = 1.$

3. $\frac{x^2}{\frac{3}{4}} + \frac{(y-\frac{1}{2})^2}{1} = 1.$

4. $\frac{(x-6)^2}{8} + \frac{(y-4)^2}{12} = 1.$

5. $x^2 + 3y^2 + 6x + 6 = 0.$

6. $x^2 + 2y^2 - 2x - 4y = 1.$

7. $8x^2 + y^2 - 16x + 2y = 0.$

8. $2x^2 + y^2 + 8x + 4y = 0.$

9. $4x^2 + y^2 = 4cx.$

10. $3x^2 + 5y^2 = 10cy.$

11. $7x^2 + 2y^2 - 28x + 4y + 16 = 0.$

12. $4x^2 + 5y^2 + 16x - 20y + 31 = 0.$

Ans. $\frac{(x+2)^2}{\frac{5}{4}} + \frac{(y-2)^2}{1} = 1.$

13. $7x^2 + 8y^2 - 28x + 80y + 172 = 0.$

14. $7x^2 + 2y^2 - 28x + 4y + 16 = 0.$

Ans. $\frac{(x-2)^2}{2} + \frac{(y+1)^2}{7} = 1.$

15. $3x^2 + 7y^2 - 12x + 28y + 19 = 0.$

16. $45x^2 + 63y^2 - 120x + 84y = 207.$

Ans. $\frac{(x-\frac{4}{3})^2}{7} + \frac{(y+\frac{3}{5})^2}{5} = 1.$

Trace the given curves, and find their points of intersection.

17. $y^2 = 4x$, $16x^2 + y^2 = 128x - 240.$ Ans. $(4, \pm 4)$, $(\frac{15}{4}, \pm \sqrt{15})$.

18. $x^2 = 4y$, $9x^2 + 4y^2 - 72x - 8y + 112 = 0.$ Ans. $(2, 1)$, $(4, 4)$.

19. $3x^2 = 20x - 4y - 24$, $(x-2)^2 + 2(y-1)^2 = 6.$

Ans. $(4, 2)$ three times, $(\frac{4}{3}, -\frac{2}{3})$.

20. $x^2 + 2y^2 = 6$, $5x^2 + 12y^2 = 16x.$

21. A circle touches the circles $x^2 + y^2 = 4$, $x^2 + y^2 - 12x = 64$. Find the equation of the locus of its center, and draw the figure.

Ans. $3x^2 + 4y^2 - 18x = 81.$

III. THE HYPERBOLA

56. Hyperbola referred to its axes: first standard form.

By § 45, the hyperbola is *the conic section for which $e > 1$* .

The derivation of equation (1), § 52, does not depend upon the fact that $e < 1$; we therefore conclude at once that, with the axes chosen as in § 52, the equation of the hyperbola is

$$(1) \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

But, since now $e > 1$, the constant $a^2(1 - e^2)$ is negative; therefore to make b real, we set

$$(2) \quad b^2 = a^2(e^2 - 1),$$

and write (1) in the form

$$(3) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

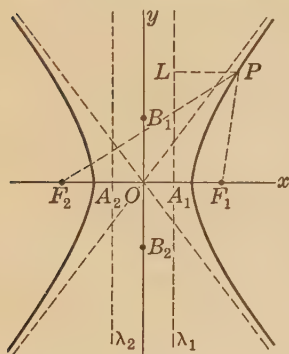


FIG. 58

This equation shows that:

(a) The curve is symmetric with respect to both axes. Hence there are two foci, the points $(\pm ae, 0)$, and two directrices, the lines $x = \pm \frac{a}{e}$.

(b) The intercepts on Ox are $(\pm a, 0)$, those on Oy are imaginary.

(c) The equation when solved for y has the form

$$(4) \quad y = \pm \frac{b}{a} \sqrt{x^2 - a^2},$$

which shows that y is imaginary if and only if $x^2 < a^2$, i.e. if $-a < x < a$. The curve therefore consists of two disconnected branches, one lying to the right of the line $x = a$, the other to the left of the line $x = -a$.

(d) *The latus rectum is $\frac{2b^2}{a}$.*

As in the case of the ellipse, the lines of symmetry are sometimes spoken of as the *axes* of the curve, but unless the contrary is indicated the axes will be considered to be the *segments* A_2A_1 of length $2a$ and B_2B_1 of length $2b$: the former is the *transverse axis*, the latter the *conjugate axis*. The conjugate axis does not intersect the curve, but plays an important part in its theory. The ends A_2, A_1 of the transverse axis are the *vertices*; the point of intersection of the axes is the *center*.

It appears from (2) that the *distance from the center to the foci* is

$$(5) \quad ae = \sqrt{a^2 + b^2}.$$

It should also be noted that in the case of the hyperbola b may be *greater than, equal to, or less than* a , according to the value of e .

It is easily seen that the equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

represents a *hyperbola with transverse axis in Oy*.

57. Asymptotes. When the standard hyperbola

$$(1) \quad b^2x^2 - a^2y^2 = a^2b^2$$

and the straight line

$$(2) \quad bx - ay = 0$$

are drawn on the same axes, the figure indicates that the hyperbola approaches the straight line more and more closely as the distance from

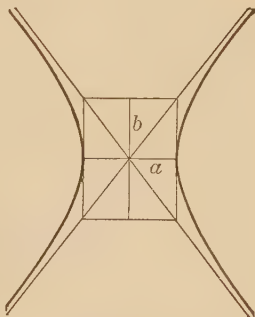


FIG. 59

the center increases, but without ever reaching the line. To prove this, let $P : (x_1, y_1)$ be a point on the hyperbola in the first or third quadrant. The distance from P to the line (2) is (§ 34)

$$(3) \quad d = \frac{bx_1 - ay_1}{\sqrt{a^2 + b^2}}.$$

Since P is on the curve, its coördinates satisfy (1):

$$b^2x_1^2 - a^2y_1^2 = a^2b^2,$$

or

$$(bx_1 - ay_1)(bx_1 + ay_1) = a^2b^2,$$

whence

$$bx_1 - ay_1 = \frac{a^2b^2}{bx_1 + ay_1}.$$

Substituting this value of $bx_1 - ay_1$ in (3), we find

$$d = \frac{a^2b^2}{\sqrt{a^2 + b^2}} \cdot \frac{1}{bx_1 + ay_1}.$$

Evidently d can never equal 0, but as P recedes, so that x_1, y_1 both increase indefinitely, d becomes smaller and smaller, approaching the limit 0.

A similar result is easily established for the line

$$(4) \quad bx + ay = 0$$

when P lies in the second or fourth quadrant.

The lines (2) and (4), or as usually written,

$$(5) \quad y = \pm \frac{b}{a} x,$$

are called the *asymptotes* * of the hyperbola. They are of great importance both in tracing the curve and in studying its properties.

*A general definition of this term will be given in § 92.

From (5) we derive the following result, which gives a convenient method for drawing the asymptotes of any hyperbola whose axes are given:

The asymptotes of a hyperbola are the diagonal lines of the rectangle whose center is the center of the curve and whose sides are parallel and equal to the axes of the curve.

To trace a hyperbola whose equation is in a standard form, we *plot the vertices and the ends of the latera recta*, and *draw the asymptotes*. These data suffice for a fairly satisfactory sketch; if greater accuracy is required, more points must be plotted.

58. Equilateral, or rectangular, hyperbola. The hyperbola for which a and b are equal is called, on account of the equality of the semi-axes, the *equilateral* hyperbola. Since the rectangle of Fig. 59 is in this case a square, the asymptotes of the equilateral hyperbola are at right angles: for this reason it is also called the *rectangular* hyperbola. Since $ae = \sqrt{a^2 + b^2}$, the eccentricity of the equilateral hyperbola is $e = \sqrt{2}$.

When the hyperbola is equilateral, equation (3) of § 56 evidently assumes the form

$$x^2 - y^2 = a^2.$$

EXERCISES

In the following cases, locate the center, vertices, foci, and ends of the latera recta, draw the asymptotes, and trace the curve. Determine the eccentricity and write the equations of the directrices and asymptotes.

1. $\frac{x^2}{4} - \frac{y^2}{2} = 1.$

2. $\frac{x^2}{4} - \frac{y^2}{5} = 1.$

3. $\frac{y^2}{9} - \frac{x^2}{16} = 1.$

4. $\frac{y^2}{6} - \frac{x^2}{3} = 1.$

5. $\frac{x^2}{16} - \frac{y^2}{1} = 1.$

6. $\frac{y^2}{25} - \frac{x^2}{4} = 1.$

7. $3x^2 - 4y^2 = 4.$

8. $2x^2 - 4y^2 + 1 = 0.$

Find the equations of the following hyperbolas, assuming the center at O and transverse axis along Ox .

9. Latus rectum 36, distance between foci 24. *Ans.* $3x^2 - y^2 = 108$.

10. Eccentricity $2\sqrt{2}$, latus rectum 6. *Ans.* $49x^2 - 7y^2 = 9$.

11. Distance between foci 6, distance between directrices 4.
Ans. $x^2 - 2y^2 = 6$.

12. Passing through $(2, 1)$, $(4, 3)$. *Ans.* $2x^2 - 3y^2 = 5$.

13. Latus rectum $\frac{4}{3}$, slope of asymptotes $\pm \frac{1}{3}$. *Ans.* $x^2 - 9y^2 = 36$.

14. Foci $(\pm 4, 0)$, slope of asymptotes ± 3 . *Ans.* $45x^2 - 5y^2 = 72$.

15. Latus rectum 18, distance between directrices 3.
Ans. $3x^2 - y^2 = 27$.

16. Latus rectum $\frac{9}{2}$, distance between directrices $\frac{3}{2}$.
Ans. $9x^2 - 16y^2 = 144$.

17. Distance between directrices 1, the rectangle on the axes of area 2.
Ans. $2x^2 - 2y^2 = 1$.

18. Distance between directrices $\frac{1}{15}\sqrt{30}$, passing through $(1, 2)$.
Ans. $5x^2 - y^2 = 1$; $24x^2 - y^2 = 20$.

19. Given a hyperbola with its axes, draw the asymptotes and find the foci geometrically.

20. Given a hyperbola with its axes, find the directrices.

21. Prove analytically that a line parallel to an asymptote of a hyperbola intersects the curve in one and only one point.

22. Draw the graph of $\sec \theta$ as a function of $\tan \theta$.

23. Prove analytically that the product of the distances of any point of a hyperbola from its asymptotes is constant.

59. Another definition of the hyperbola. A property of the hyperbola frequently used as a *definition* of the curve is as follows (cf. § 53):

A hyperbola is the locus of a point which moves so that the difference of its distances from two fixed points is constant. The fixed points are the *foci* and the constant difference is the *transverse axis*.

That is, in Fig. 58,

$$(1) \quad F_2P - F_1P = 2a.$$

The proof is very similar to that of § 53, and may be carried out by the student.

EXERCISES

1. A point moves so that the difference of its distances from $(3, 0)$, $(-3, 0)$ is 2. Find the equation of its locus in two ways. (§§ 21, 59.)
2. A point moves so that the difference of its distances from $(0, 4)$, $(0, -4)$ is 6. Find the equation of its locus in two ways.
3. A point moves so that the difference of its distances from $(ae, 0)$, $(-ae, 0)$ is $2a$. Find the equation of its locus by the general method. (§ 21.)
4. A circle touches the circle $x^2 + y^2 + 2cx = 0$ and passes through $(c, 0)$. Find the locus of its center. *Ans.* $12x^2 - 4y^2 = 3c^2$.
5. A circle passes through a given point and touches a given circle. Find the locus of its center, for all possible cases. (§§ 53, 59.)
6. A moving circle is tangent externally to the two fixed circles $(x - 2c)^2 + y^2 = c^2$, $(x + 2c)^2 + y^2 = 4c^2$. Find the locus of its center. *Ans.* $60x^2 - 4y^2 = 15c^2$.
7. Given the foci and transverse axis of a hyperbola, show how to construct points of the curve by ruler and compass.
8. Given the foci and one point of a hyperbola, construct the axes.
9. Given the foci and asymptotes of a hyperbola, find the vertices.
10. The sound of a gun and the ring of the ball on the target are heard simultaneously at a point P . Construct the locus of P .
11. Prove equation (1), § 59.
12. A moving circle is tangent to two fixed circles. Find the locus of the center of the moving circle, for all possible cases. (§§ 53, 59.)

60. Other standard forms. Formulas for the hyperbola analogous to those of § 54 for the ellipse are as follows:

The equation of a hyperbola with center at (h, k) is, if the transverse axis is parallel to Ox ,

$$(1) \quad \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1;$$

if the transverse axis is parallel to Oy ,

$$(2) \quad \frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1.$$

The proof is left to the student.

61. General equation. The equation

$$(1) \quad Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

where A and C have *opposite* signs, can evidently be reduced, by completing the squares in x and y , to one of the forms (1), (2) of § 60. The only exceptional case is the one in which, when the left member has been expressed as the difference of two squares, the right member reduces to 0:

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 0.$$

This equation can be factored, and therefore represents two straight lines, intersecting at (h, k) . Hence the

THEOREM: *An equation of the second degree in which the xy -term is missing and the coefficients of x^2 and y^2 have unlike signs represents a hyperbola with its axes parallel to the co-ordinate axes (exceptionally, two intersecting lines).*

Example: Trace the curve

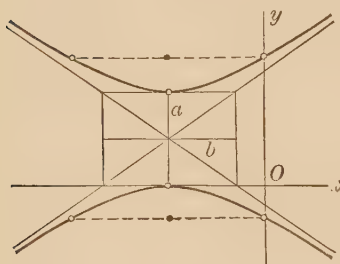


FIG. 60

$$x^2 - 2y^2 + 4x + 4y + 4 = 0.$$

Completing the squares in x and y , we get

$$(x + 2)^2 - 2(y - 1)^2 = -2,$$

or, dividing by -2 ,

$$\frac{(y - 1)^2}{1} - \frac{(x + 2)^2}{2} = 1.$$

This is a hyperbola with center at $(-2, 1)$ and transverse axis parallel to Oy ; the semi-axes are $a = 1$, $b = \sqrt{2}$. The vertices are at the distance 1, the foci at the distance $\sqrt{a^2 + b^2} = \sqrt{3}$, above and below the center. The latus rectum is $\frac{2b^2}{a} = 4$, so that the ends of the latera recta are

at the distance 2 to right and left of the foci. The asymptotes are the diagonals of the rectangle of sides $2a$, $2b$.

EXERCISES

In the following cases, locate the center, vertices, foci, and ends of the latera recta, draw the asymptotes, and trace the curve.

$$1. \frac{(x+3)^2}{4} - \frac{(y-1)^2}{8} = 1.$$

$$2. \frac{(x-1)^2}{3} - \frac{y^2}{1} = 1.$$

$$3. \frac{(y+\frac{3}{2})^2}{1} - \frac{(x+\frac{1}{2})^2}{\frac{1}{4}} = 1.$$

$$4. \frac{(y-6)^2}{36} - \frac{(x-5)^2}{4} = 1.$$

$$5. 5x^2 - 4y^2 - 20x - 24y - 36 = 0.$$

$$\text{Ans. } \frac{(x-2)^2}{4} - \frac{(y+3)^2}{5} = 1.$$

$$6. 9x^2 - 16y^2 = 36x - 96y - 36.$$

$$\text{Ans. } \frac{(y-3)^2}{9} - \frac{(x-2)^2}{16} = 1.$$

$$7. x^2 - y^2 + 6x + 2y + 10 = 0.$$

$$\text{Ans. } \frac{(y-1)^2}{2} - \frac{(x+3)^2}{2} = 1.$$

$$8. 2x^2 - 2y^2 + x - y - 1 = 0.$$

$$\text{Ans. } \frac{(x+\frac{1}{4})^2}{\frac{1}{2}} - \frac{(y+\frac{1}{2})^2}{\frac{1}{2}} = 1.$$

$$9. x^2 - 2y^2 + ay = 0.$$

$$10. 4x^2 - y^2 - 2ax = 0.$$

$$11. 4y^2 = x^2 - 3x + 4.$$

$$12. x^2 - 4y^2 + 4ax - 8ay = 0.$$

$$13. 3x^2 - 4y^2 + 6x + 6y = 0.$$

$$14. 4x^2 - 4y^2 + 4x - 8y = 1.$$

$$15. x^2 - 4y^2 + 2x + 8y = 3.$$

$$16. 16x^2 - y^2 - 4x + y - 6 = 0.$$

17. Show that if $C = -A$, equation (1) of § 61 represents an equilateral hyperbola.

Trace the following curves, and find their points of intersection.

$$18. x^2 - 9y^2 - 2x - 18y - 17 = 0, \quad x + 3y + 2 = 0.$$

$$19. 4x^2 - y^2 - 8x - 2y - 1 = 0, \quad 2x - y = 3.$$

$$20. x^2 - y^2 + 3x - y + 8 = 0, \quad x^2 - y^2 + 4x - 4y + 16 = 0.$$

$$\text{Ans. } (-2, 2), (1, 3).$$

$$21. x^2 - y^2 + 5x + 3y + 2 = 0, \quad x^2 - y^2 + 2x = 4. \quad \text{Ans. } (-4, 2).$$

$$22. x^2 - y^2 + 2x + 4y = 7, \quad x^2 - y^2 + 4x + 4y = 1.$$

$$\text{Ans. } (-3, 2) \text{ twice.}$$

23. A point moves so that the product of its distances from two intersecting lines is constant. Prove that its locus is a hyperbola having the given lines as asymptotes. (Take the lines $y = \pm mx$. Is the proof general?)

CHAPTER VII

COÖRDINATE TRANSFORMATIONS

62. Translation of axes. We have had many illustrations of the fact that a problem may be greatly simplified by taking the axes in a convenient position. It may happen, however, that the position of the axes is predetermined by the statement of the problem; it is then desirable that we have methods for shifting the axes to some more suitable position. Such methods will now be developed.

Consider first the *translation* of axes, in which the axes are moved parallel to their original positions. Let Ox, Oy be the original axes, $O'x', O'y'$ the new, and suppose the new origin O' to be the point (h, k) referred to the old axes. If we denote by (x, y) the coördinates of any point P in the original system, by (x', y') the coördinates of the same point in the new system, then it is obvious from the figure

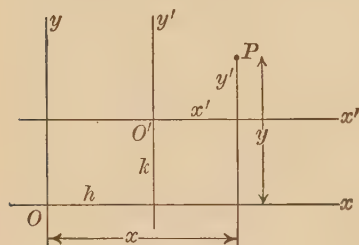


FIG. 61

that the two sets of coördinates are connected by the formulas

$$(1) \quad \begin{cases} x = x' + h, \\ y = y' + k. \end{cases}$$

An important problem is, being given the equation of a curve referred to any set of axes, to derive its equation referred to parallel axes (i.e. axes parallel to the original ones) through a point (h, k) as new origin. To do this, we have merely to replace x and y in the equation of the curve by their values as given by (1).

Example: Obtain the equation of the ellipse (p. 90)

$$9x^2 + 4y^2 - 36x + 8y + 31 = 0$$

referred to parallel axes through $(2, -1)$ as new origin.

Substitute

$$x = x' + 2, \quad y = y' - 1:$$

$$9x'^2 + 36x' + 36 + 4y'^2 - 8y' + 4 - 36x' - 72 + 8y' - 8 + 31 = 0,$$

or

$$9x'^2 + 4y'^2 = 9.$$

63. Rotation of axes. Let Ox, Oy and Ox', Oy' be two pairs of cartesian axes with the same origin, and denote by ϕ the angle through which the first pair must be rotated to come to coincidence with the second, the angle ϕ being considered positive when measured counterclockwise. Let P have the coördinates x, y in the first system and x', y' in the second, so that

$$\begin{aligned} OM &= x, & MP &= y, \\ OM' &= x', & M'P &= y'. \end{aligned}$$

Now

$$\begin{aligned} OM &= ON - MN \\ &= ON - QM'; \\ MP &= MQ + QP \\ &= NM' + QP. \end{aligned}$$

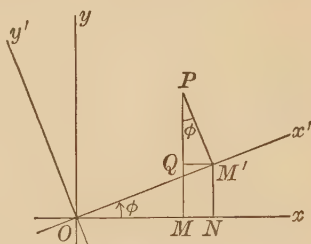


FIG. 62

We thus have the formulas

$$(1) \quad \begin{cases} x = x' \cos \phi - y' \sin \phi, \\ y = x' \sin \phi + y' \cos \phi. \end{cases}$$

Both in translation and in rotation of axes the original coördinates x, y are replaced by expressions of the *first degree* in x', y' : from this it follows that *the degree of an equation is unchanged by translation or rotation of axes.*

It will be seen that the formulas of this and the preceding article taken together enable us to change the axes from any position Ox , Oy to any other position $O'x'$, $O'y'$. For we can first translate the axes to a parallel position with O' as new origin, and then rotate them through the angle necessary to bring them to the desired position.

Example: Rotate the axes through an angle ϕ such that the term in y will disappear from the equation

$$(2) \quad 4x + 3y = 5.$$

Substituting in (2) the values of x and y from (1), we get

$$4(x' \cos \phi - y' \sin \phi) + 3(x' \sin \phi + y' \cos \phi) = 5,$$

or

$$(3) \quad (4 \cos \phi + 3 \sin \phi)x' + (-4 \sin \phi + 3 \cos \phi)y' = 5.$$

To make y' drop out, we equate its coefficient to 0:

$$-4 \sin \phi + 3 \cos \phi = 0,$$

whence

$$\tan \phi = \frac{3}{4}, \quad \cos \phi = \frac{4}{5}, \quad \sin \phi = \frac{3}{5}.$$

These values substituted in (3) give

$$\left(\frac{16}{5} + \frac{9}{5}\right)x' = 5, \quad x' = 1.$$

EXERCISES

1. Find the coördinates of the points $(3, 4)$, $(5, -3)$, $(0, 2)$ referred to parallel axes through $(3, -1)$. Check by plotting.
2. Find the coördinates of the points $(1, -3)$, $(6, 2)$, $(0, -5)$ referred to parallel axes through $(5, -3)$. Check by plotting.
3. In Exs. 12, 14, 16, p. 91, obtain the equation of the ellipse referred to parallel axes through its center.
4. In Exs. 6, 7, 8, p. 99, obtain the equation of the hyperbola referred to parallel axes through its center.
5. By translation of axes, remove the terms of first degree from the equation $x^2 - 4y^2 + x - 4y = 0$.

6. By translation of axes, remove the term in x and the constant term from the equation $x^2 + y^2 + 2x - 3 = 0$.

7. By rotating the axes through 45° , prove that the equation $2xy = a^2$ represents a rectangular hyperbola.

8. By rotating the axes through an angle $\phi = \arctan \frac{4}{3}$, prove that the equation $(4x - 3y)^2 = y$ represents a parabola.

64. Expansion and contraction in one dimension. The transformations above discussed do not change the size, shape, or position of any figure in the plane—they merely move the axes. We consider now a transformation of different character.

Leaving x unchanged, let us introduce a new y -coördinate that is *proportional* to the original one: i.e. put

$$(1) \quad x' = x, \quad y' = ky.$$

This evidently transforms the point $(1, 1)$ into the point $(1, k)$; etc. Thus the vertical dimensions of any figure are *expanded* (if $k > 1$) or *contracted* (if $k < 1$) to k times their original size, the horizontal dimensions being unchanged: a *stretch* or a *compression* respectively. This of course changes the shape of the figure.

Similarly, we may stretch or compress the figure horizontally:

$$x' = kx, \quad y' = y.$$

In tracing a curve, this transformation means merely that we have adopted different scales in the two directions, the y -scale, in (1), being k times the x -scale. The usefulness of this device, on occasion, was mentioned in § 4; see also Exs. 37–40, p. 6, and Exs. 22–25, p. 11. The principal application of the transformation arises in drawing figures where the values of y are so large, or so small, in comparison with the values of x that a suitable scale in one direction will run the sketch off the paper, or reduce it to microscopic dimensions, in the other.

65. Similar figures. Two geometric figures are said to be *similar*, or to have *the same shape*, if they can be so placed, with reference to a point C , that every line joining C to a point P of the first figure meets the second figure in a point Q , and vice versa; and if, further, the ratio of CQ to CP is constant for all positions of the line:

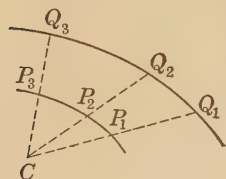


FIG. 63

$$\frac{CQ_1}{CP_1} = \frac{CQ_2}{CP_2} = \frac{CQ_3}{CP_3} = \dots = k.$$

When the two figures actually are so placed, they are said to be *similarly placed*, or placed *in perspective*; the point C is called the *center of similitude*.

Similar figures occur frequently in geometry. For example, all circles are similar; two triangles are similar if the sides of one are proportional to the sides of the other, or—what amounts to the same thing—if the angles of one are equal to the angles of the other; etc.

66. Transformation of similitude. Let us introduce two new coördinates each bearing the *same ratio* to the original ones:

$$(1) \quad x' = kx, \quad y' = ky.$$

This evidently means that all dimensions of any figure are increased, if $k > 1$, or decreased, if $k < 1$, in the ratio k : the figure expands or contracts in size, but its shape is unchanged. That is, the original figure and the new one are *similar*. They are also similarly placed, with O as the center of similitude.

The transformation just described, called a *transformation of similitude*, has important applications in the study of similar curves.

67. Conics having the same eccentricity. We are now in position to prove the

THEOREM: *Any two conics having the same eccentricity are similar; and conversely.*

COROLLARY: *All parabolas are similar.*

Consider the equation of the standard ellipse in the primary form (1), § 52:

$$(1) \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1,$$

and suppose that e is given, but that a may be assigned any value at pleasure. In (1), if we put

$$(2) \quad x = ax', \quad y = ay',$$

the equation reduces at once to

$$(3) \quad x'^2 + \frac{y'^2}{1 - e^2} = 1,$$

which is a single, unique ellipse: viz. the ellipse with eccentricity e and semi-major axis 1. Since the transformation

(2) is of the class (1), § 66, with $k = \frac{1}{a}$, it has not changed

the shape of the figure: i.e. all the ellipses represented by (1), for different values of a , are similar to (3) and therefore to each other. It will appear presently that, by combined translation and rotation of axes, the equation of any conic can be reduced to the first standard form for that conic. Since these transformations do not change the shape of the curve, the proof of the theorem is complete for the ellipse. Since the proof does not depend on the fact that $e < 1$, it applies at once to the hyperbola.

The corollary may be proved by the student; proof of the converse will be deferred (Ex. 14, p. 165).

68. Reflections. When two curves are symmetric to each other with respect to a given line, each is said to be the *image*, or *reflection*, of the other with respect to that line. A transformation which replaces a curve by its image with respect to some line is called a *reflection in that line*.

The most important reflections are:

- (a) *Reflection in the y-axis.*
- (b) *Reflection in the x-axis.*
- (c) *Reflection in the line $y = x$.*

To effect the transformation (a)—that is, to replace a curve by its image in Oy —we have merely to *reverse the sense of the x-axis*. Obviously, to do this we merely *change x to $-x$* .

To effect (b), *change y to $-y$* .

To effect (c), *interchange x and y* . To see this, note that this change transforms any point (h, k) into the point (k, h) , and it is easily seen either geometrically or analytically that these points are symmetrically placed with reference to the line $y = x$.

Examples: (a) The parabola $y^2 = -x$ is the image of the parabola $y^2 = x$ with respect to the y -axis.

(b) The parabola $x^2 = 4ay$ is the image of the parabola $y^2 = 4ax$ with respect to the line $y = x$.

EXERCISES

1. Plot the curve $x^2 + y^2 = a^2$ (a) with equal scales in x and y ; (b) with the same x -scale as before but the y -scale half as large; (c) with the y -scale three times as large.

2. Solve Ex. 1 for the hyperbola $x^2 - y^2 = a^2$.

3. Solve Ex. 1 for the parabola $y^2 = 4ax$.

4. Solve Ex. 1 for the ellipse $4x^2 + y^2 = 4ay$.

5. Draw the curve $y = x^2$, in the interval $0 \leq x \leq 10$, (a) with equal scales; (b) with a suitable expansion or compression in one direction.

6. Solve Ex. 5 for the curve $y = x^3$, $0 \leq x \leq 5$.
 7. Plot the ellipse $x^2 + 100y^2 = 1$.
 8. Plot the hyperbola $400x^2 - y^2 = 16$.
 9. Prove that any ellipse can be transformed into a circle by a suitably chosen stretch or compression. (§ 64.)
 10. Prove the corollary in § 67. (Apply the transformation (2), § 67, to the equation $y^2 = 4ax$.)
 11. Show that the curves $y = x^3$, $4y = x^3$, $y = 9x^3$ all have the same shape. (Apply the transformation (2), § 67, to the equation $a^2y = x^3$.) Plot the three curves on the same axes, choosing scales so that the three will coincide.
 12. Solve Ex. 11 for the curves $y^2 = x^3$, $3y^2 = x^3$, $y^2 = 4x^3$.
- Verify the following statements by tracing the curves.
13. $x^2 = 4a^2 - 4ay$ is the reflection of $x^2 = 4a^2 + 4ay$ in the x -axis.
 14. $x^2 + y^2 + 2ax = 0$ is the reflection of $x^2 + y^2 - 2ax = 0$ in the y -axis.
 15. $x^2 = 4a^2 + 4ay$ is the reflection of $y^2 = 4a^2 + 4ax$ in the line $y = x$.
 16. $y = x^{1/3}$ is the reflection of $y = x^3$ in the line $y = x$.

CHAPTER VIII

THE GENERAL EQUATION OF SECOND DEGREE

69. Removal of the product term. The most general equation of the second degree has the form

$$(1) \quad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The results of §§ 50, 55, 61 show that when $B = 0$ this equation (if it has any locus at all) always represents a conic section, and it is reasonable to expect that the same will be true in case $B \neq 0$. This will be an established fact if we can show that by a translation or rotation of axes (§§ 62, 63) the term in xy can always be removed from

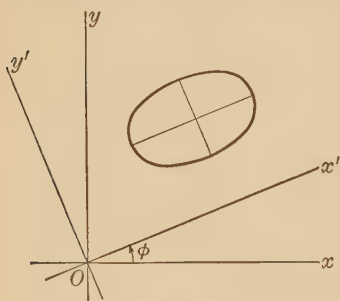


FIG. 64

equation (1). The transformation to be used is suggested by the following line of thought: If a conic lies with its axes inclined to the coördinate axes, as in Fig. 64, we can always rotate the axes through an angle ϕ (§ 63) to the position Ox' , Oy' parallel to the axes of the curve, and we know

that when this has been done the equation of the conic must be free of the product term. Hence, substitute for x and y in (1) the values given by (1), § 63:

$$(2) \quad \begin{aligned} & A(x' \cos \phi - y' \sin \phi)^2 \\ & + B(x' \cos \phi - y' \sin \phi)(x' \sin \phi + y' \cos \phi) \\ & + C(x' \sin \phi + y' \cos \phi)^2 + D(x' \cos \phi - y' \sin \phi) \\ & + E(x' \sin \phi + y' \cos \phi) + F = 0. \end{aligned}$$

This equation will not contain the term in $x'y'$ if we equate the coefficient of that term to 0 and determine ϕ from the resulting formula. Upon expanding and collecting terms, it appears that this gives

$2A \sin \phi \cos \phi - 2C \sin \phi \cos \phi - B \cos^2 \phi + B \sin^2 \phi = 0$,
which may be written

$$(3) \quad (A - C) 2 \sin \phi \cos \phi = B(\cos^2 \phi - \sin^2 \phi).$$

By the double-angle formulas of trigonometry, (3) becomes

$$(4) \quad (A - C) \sin 2\phi = B \cos 2\phi,$$

or *

$$(5) \quad \tan 2\phi = \frac{B}{A - C}.$$

Now for every possible value of $\tan 2\phi$, from $-\infty$ to $+\infty$, there is a value of 2ϕ between 0 and π , hence a value of ϕ between 0 and $\frac{1}{2}\pi$, so that it is always possible to determine a positive acute angle ϕ satisfying (4). If the curve (1) be referred to axes making this angle with the original ones, the resulting equation will be free of the product term, and its locus must be a conic. Hence we have the

THEOREM: *Every equation of the second degree (if it has a locus) represents a conic section, whose axes are inclined to the coördinate axes at the positive acute angle † ϕ given by formula (5), or, if $A = C$, $B \neq 0$, at an angle of 45° .*

Thus to trace the locus of any equation of the form (1) the first step is to determine $\tan 2\phi$ by (5), and then obtain $\cos \phi$ and $\sin \phi$ by the trigonometric formulas

* Formula (5) fails if $A = C$. But if $A = C$, $B \neq 0$, then $\phi = 45^\circ$; for (4) gives $\cos 2\phi = 0$, $2\phi = 90^\circ$. If $A = C$, $B = 0$, then (4) is true for all values of ϕ , which is to be expected. Why?

† There are of course infinitely many other values of ϕ , differing from this one by multiples of $\frac{1}{2}\pi$, but for definiteness we will agree to choose always the positive acute angle.

$$(6) \quad \cos \phi = \sqrt{\frac{1 + \cos 2\phi}{2}}, \quad \sin \phi = \sqrt{\frac{1 - \cos 2\phi}{2}}.$$

The values of $\cos \phi$ and $\sin \phi$, when substituted in (1), § 63, give us the expressions for x and y in terms of the new coördinates. Substituting these expressions in the original equation, we have the equation of the curve referred to the new axes. This equation will contain no xy -term, so that the curve may be traced by the methods developed in §§ 50, 55, 61.

Example: Trace the curve

$$(7) \quad 9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0.$$

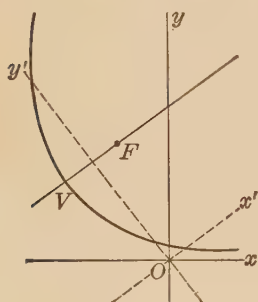


FIG. 65

Here we have

$$\tan 2\phi = \frac{-24}{9 - 16} = \frac{24}{7},$$

whence

$$\cos 2\phi = \frac{7}{25},$$

and by (6),

$$\cos \phi = \frac{4}{5}, \quad \sin \phi = \frac{3}{5}.$$

Thus by § 63,

$$x = \frac{1}{5}(4x' - 3y'), \quad y = \frac{1}{5}(3x' + 4y').$$

In terms of the new coördinates equation (7) becomes

$$25y'^2 - 75x' - 70y' + 19 = 0,$$

or, in the standard form,

$$(y' - \frac{7}{5})^2 = 3(x' + \frac{2}{5}).$$

This is a parabola with axis parallel to Ox' , opening in the positive direction, and with its vertex at the point $(-\frac{2}{5}, \frac{7}{5})$ referred to the new axes. The x' -axis has a slope $\tan \phi = \frac{3}{4}$; after drawing the new axes the curve is traced as in Fig. 65.

70. Invariants. In (2), § 69, let A' , B' , C' denote the coefficients of x'^2 , $x'y'$, y'^2 respectively. Expanding parentheses and collecting terms in (2), we find

$$(1) \quad A' = A \cos^2 \phi + B \cos \phi \sin \phi + C \sin^2 \phi,$$

$$(2) \quad B' = -2A \cos \phi \sin \phi + B \cos^2 \phi - B \sin^2 \phi \\ + 2C \cos \phi \sin \phi,$$

$$(3) \quad C' = A \sin^2 \phi - B \cos \phi \sin \phi + C \cos^2 \phi.$$

If we square (2) and subtract 4 times the product of (1) and (3), member by member—i.e. if we form the quantity $B'^2 - 4A'C'$ —it turns out that

$$B'^2 - 4A'C' = (B^2 - 4AC)(\cos^4 \phi + 2 \cos^2 \phi \sin^2 \phi \\ + \sin^4 \phi) \\ = (B^2 - 4AC)(\cos^2 \phi + \sin^2 \phi)^2,$$

or

$$(4) \quad B'^2 - 4A'C' = B^2 - 4AC.$$

That is, any rotation of axes leaves the value of the quantity $B^2 - 4AC$ *unchanged*. It is easily seen that the same is true of any translation of axes. For this reason the quantity $B^2 - 4AC$ is called an *invariant* of the equation.

By merely adding equations (1) and (3) member by member, we find that *the quantity $A + C$ is likewise an invariant*.

Finally, it can be shown that there is a third invariant quantity, usually denoted by Δ , and called the *discriminant* of the equation of second degree. This expression is most conveniently written in determinant form:

$$\Delta = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix}.$$

The importance of the discriminant lies chiefly in the following

THEOREM: *An equation of the second degree (if it has a locus) represents: if $\Delta \neq 0$, a typical conic—an ellipse (including the circle), a parabola, or a hyperbola; if $\Delta = 0$, a degenerate conic.*

The proof of this theorem will be omitted.

71. Test for species of a conic. When the xy -term has been removed, B' becomes 0, and the invariant relation (4) of § 70 reduces to

$$(1) \quad -4A'C' = B^2 - 4AC.$$

(a) If $B^2 - 4AC < 0$, then by (1), A' and C' must have the same sign.

(b) If $B^2 - 4AC = 0$, then either A' or C' must be 0.

(c) If $B^2 - 4AC > 0$, then A' and C' must have opposite signs.

Combining this result with the theorems of §§ 50, 55, 61, 70, we are able to classify the loci of all equations of second degree as follows.

If $\Delta \neq 0$, and

(a) *if $B^2 - 4AC < 0$, an ellipse (or no locus);*

(b) *if $B^2 - 4AC = 0$, a parabola;*

(c) *if $B^2 - 4AC > 0$, a hyperbola.*

If $\Delta = 0$, and

(d) *if $B^2 - 4AC < 0$, a point;*

(e) *if $B^2 - 4AC = 0$, two parallel or coincident lines (or no locus);*

(f) *if $B^2 - 4AC > 0$, two intersecting lines.*

The condition $B^2 - 4AC = 0$ evidently means that $Ax^2 + Bxy + Cy^2$ is a perfect square. Hence an equation of the second degree represents a parabola—exceptionally, one of the forms under (e)—*if and only if the terms of the second degree form a perfect square.*

EXERCISES

Determine the species of conic represented by the given equation; remove the xy -term by rotation of axes, reduce the resulting equation to a standard form, and trace the curve on the new axes.

1. $xy = 2$.
2. $8xy + 1 = 0$.
3. $5x^2 - 6xy + 5y^2 = 8$.
4. $2x^2 + 3xy + 2y^2 = 7$.
Ans. $x^2 - 4y^2 = 16$.
5. $3x^2 - 10xy + 3y^2 + 32 = 0$.
Ans. $2x^2 - y^2 = 2$.
6. $7x^2 + 12xy - 2y^2 = 10$.
Ans. $x^2 + 8y^2 = 16$.
7. $41x^2 - 84xy + 76y^2 = 208$.
Ans. $2x^2 + y^2 = 2$.
8. $19x^2 + 6xy + 11y^2 = 20$.
Ans. $x^2 - 4y^2 = 8$.
9. $4xy - 3y^2 = 8$.
Ans. $11x^2 + y^2 = 44$.
10. $x^2 + 3xy + 5y^2 = 22$.
Ans. $x^2 - 2x = 6y$.
11. $9x^2 + 24xy + 16y^2 + 90x - 130y = 0$.
Ans. $x^2 + 4y + 1 = 0$.
12. $3x^2 + 2\sqrt{3}xy + y^2 - 8x + 8\sqrt{3}y + 4 = 0$.
Ans. $4y^2 - (x - 1)^2 = 1$.
13. $11x^2 - 24xy + 4y^2 + 6x + 8y = 10$.
Ans. $4y^2 - (x - 1)^2 = 1$.
14. Represent $\tan(\theta + 45^\circ)$ as a function of $\tan \theta$. (Expand by the addition formula; put $\tan \theta = x$, $\tan(\theta + 45^\circ) = y$.)
15. Draw a curve from which $\sin(\theta + 60^\circ)$ may be read off if $\sin \theta$ is given. (Note the suggestion in Ex. 14.)
16. Prove that the equation $x^{1/2} \pm y^{1/2} = \pm a^{1/2}$ represents a parabola, and trace the curve.
17. Prove that if A and C have opposite signs the equation of the second degree always represents a hyperbola. Is the converse true?
18. Prove that if the xy -term is present and either or both square terms are missing the equation of second degree always represents a hyperbola.
19. A point moves so that the square of its distance from a fixed point is proportional to the product of its distances from two perpendicular lines. (Take these lines as axes.) Prove that its locus is an ellipse, parabola, or hyperbola according as the proportionality constant is less than, equal to, or greater than 2. Investigate exceptional cases.
20. Prove that if a line is parallel to the axis of a parabola, it intersects the curve in one and only one point. (Take the equation of the parabola in the first standard form. Is the proof general?)
21. Prove that if a line is parallel to an asymptote of a hyperbola, it meets the curve in one and only one point. (Note the suggestion in Ex. 20.)
22. Prove that if the terms of first degree are missing, the equation of second degree cannot represent a parabola.

72. Composition of ordinates. Let it be required to trace a curve whose equation may be written in the form

$$(1) \quad y = u + v,$$

where u and v are any two expressions* in x . It may happen that equation (1) is complicated and the curve difficult to trace, but that the curves

$$(2) \quad y = u, \quad y = v$$

are simple and easily handled. In such a case we make use of the obvious fact that, for any given value of x ,

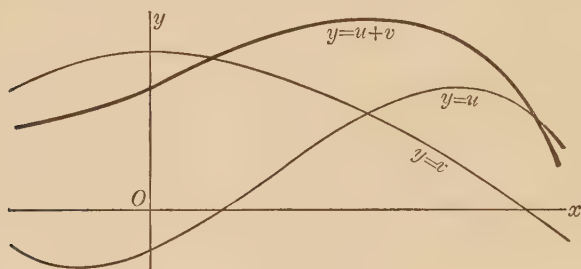


FIG. 66

the ordinate of (1) is the *sum of the ordinates* of the separate curves (2). If these two curves are traced, any number of points of (1) are easily plotted.

73. Conics traced by composition. The method of § 72 is particularly useful in tracing a conic when the xy -term is present.† We have only to *solve the equation as a quadratic in y* , which exhibits y as the sum of the ordinates of two curves, one of which is a straight line, while the other is a conic with axes parallel to the coördinate axes, and thus amenable to the methods of Chapter VI. If more convenient, we may solve for x and compound abscissas instead.

* Of any form whatever. The discussion is general.

† Many other uses for the method occur in the applications of mathematics.

Example: Trace the curve

$$x^2 - 2xy + y^2 + x - 1 = 0.$$

This curve is a parabola, since

$$B^2 - 4AC = 4 - 4 = 0.$$

Solving for y , we find

$$\begin{aligned} y &= x \pm \frac{1}{2}\sqrt{4x^2 - 4x^2 - 4x + 4} \\ &= x \pm \sqrt{1 - x}. \end{aligned}$$

The ordinates of this curve are found by adding the ordinates of the straight line

$$(1) \quad y = x$$

and the curve

$$y = \pm \sqrt{1 - x},$$

that is,

$$(2) \quad y^2 = 1 - x.$$

Equation (2) when reduced to a standard form is

$$(3) \quad y^2 = -(x - 1);$$

this is a parabola with vertex at $(1, 0)$, opening to the left. We now trace the curves (1) and (3) and locate points on the required curve by adding (algebraically) the ordinates of these two curves. For a given value of x there are two points on the required curve, with ordinates

$$\begin{aligned} y &= x + \sqrt{1 - x}, \\ y &= x - \sqrt{1 - x}. \end{aligned}$$

For instance, when $x = OM$, the two ordinates are

$$MP_1 = MR + MQ_1, \quad MP_2 = MR + MQ_2.$$

Due regard must of course be paid to the sign of each segment.

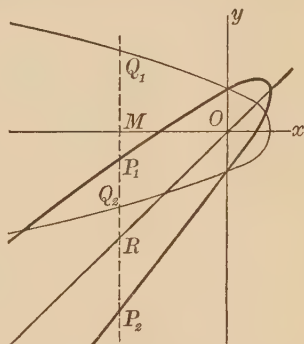


FIG. 67

On account of its complete generality, the method of § 69 furnishes a theoretically perfect tool for tracing any conic. However, it frequently leads to awkward numerical difficulties due to the cumbersome irrational values assumed by $\cos \phi$ and $\sin \phi$. By the method of composition these difficulties are greatly reduced, since the equations of the component straight line and conic will have rational coefficients whenever the original coefficients are rational. On the other hand, the method yields much less information about the given curve, since it does not immediately locate the vertices, foci, etc.*

74. Equilateral hyperbola with asymptotes parallel to the coördinate axes. The equation

$$(1) \quad Bxy + Dx + Ey + F = 0$$

represents a hyperbola (or two intersecting lines), since $B^2 - 4AC = B^2 > 0$.

Let us solve the equation for y :

$$y = -\frac{Dx + F}{Bx + E}$$

When x approaches the value $-\frac{E}{B}$, i.e. when $Bx + E$ approaches 0, y increases indefinitely. Thus the line

$$Bx + E = 0$$

is an asymptote. By solving for x , we find that the line

$$By + D = 0$$

is the other asymptote. Thus equation (1) represents an equilateral hyperbola with its asymptotes parallel to the coördinate axes.

* A third method, involving a suitably chosen translation of axes followed by use of the invariants $A + C$ and $B^2 - 4AC$, obtains the equation of the conic in its first standard form, and frequently involves less labor than the method of § 69. For a presentation of this method, see, for example, Osgood and Graustein, *Plane and Solid Analytic Geometry* (New York, Macmillan Co.)

Covering an important special case, we have the

THEOREM: *The equation*

$$xy = k \quad (k \neq 0)$$

represents a rectangular hyperbola whose asymptotes are the coördinate axes.

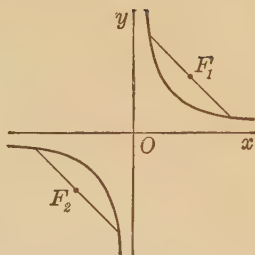


FIG. 68

EXERCISES

Trace the following curves by § 73, first applying the test of § 71.

1. $2x^2 - 2xy + y^2 = 1$.
2. $5x^2 - 4xy + y^2 = 4$.
3. $x^2 + 2xy + y^2 = 4x + 2y$.
4. $2xy - y^2 - 6x + 13 = 0$.
5. $8x^2 + 12xy + 9y^2 + 4x + 6y = 0$.
6. $x^2 - 4xy + 5y^2 - 20x + 42y + 100 = 0$.
7. $4x^2 - 4xy = 1$.
8. $3x^2 - 2xy + y^2 - 5x + 4y + 3 = 0$.
9. $3x^2 + 6xy - 9y^2 + 2x - 18y - 9 = 0$.
10. $x^2 - 6xy + 10y^2 + 8x - 23y + 10 = 0$. (Solve for x .)
11. Trace the curve of Ex. 16, p. 113, by composition of ordinates.

12. The following statements (§ 73) have not been proved: (a) when the equation of a conic is solved for y , y appears as the sum of the ordinates of a straight line and a conic; (b) the axes of this conic are parallel to the coördinate axes; (c) the equations of the component line and conic have rational coefficients whenever the original coefficients are rational. By solving (1), § 69, for y ($C \neq 0$), prove these statements.

13. By solving (1), § 69, for y , show that when $B^2 - 4AC = -4C^2$, the ordinates of the conic are the sum of ordinates of a straight line and a circle. (Cf. Ex. 1.)

14. Obtain a theorem corresponding to that of Ex. 13, when abscissas are compounded. (Cf. Ex. 10.)

In the following cases, by solving for x and y in turn, locate the asymptotes; then plot the curve by points.

15. $xy + 4 = 0$.
16. $xy = 3$.
17. $xy + 3x - 2y = 0$.
18. $xy - 4x + y = 0$.
19. $2xy + 5x - 2y + 2 = 0$.
20. $2xy + 3y + 6 = 0$.

CHAPTER IX

PROPERTIES OF THE CONICS

I. TANGENTS AND NORMALS

75. Tangents to plane curves. A straight line that intersects a curve in two or more distinct points is called a *chord*, or *secant*. The term “chord” is also used as in elementary geometry to denote a line segment terminated by two points of a curve.

Let P be a fixed point of a plane curve, and P' a neighboring point. If P' be made to approach P along the curve, the secant PP' evidently approaches, in general, a definite limiting position, the line PT in the figure. The line thus approached is called the *tangent to the curve at P* , or is said

to *touch the curve at P* . The point P is the *point of contact*.

The *slope of a curve* at any point is defined as the *slope of the tangent* at that point. Two curves intersecting in a point P are said to be *tangent at P* if they

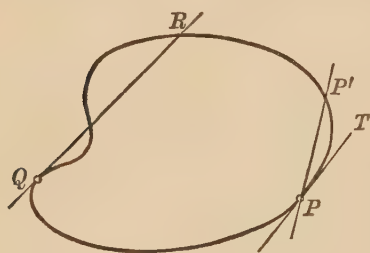


FIG. 69

have the same slope—i.e. if they have a common tangent—at that point.

In elementary geometry a tangent to a circle is usually defined as a line which intersects the circle in one point. Such a definition would not hold for curves in general, as is shown by Fig. 69, where the line QR is both a secant and a tangent. The definition given above holds in general.

76. Tangent at a given point. We proceed to develop a method of finding the tangent at any point of a plane curve, beginning with an

Example: Find the tangent to the parabola

$$(1) \quad y^2 - 3x - y + 1 = 0$$

at the point $P : (1, 2)$.

Choose a point P' on the curve near the given point, and denote the distances PR , RP' by Δx , Δy respectively, so that the coördinates of P' are $(1 + \Delta x, 2 + \Delta y)$. Since P' lies on the curve, its coördinates may be substituted for x and y in (1). This gives

$$(2 + \Delta y)^2 - 3(1 + \Delta x) - (2 + \Delta y) + 1 = 0,$$

which reduces to

$$3\Delta y + \Delta y^2 - 3\Delta x = 0.$$

Dividing through by Δx , we get

$$(2) \quad 3\frac{\Delta y}{\Delta x} + \frac{\Delta y}{\Delta x}\Delta y - 3 = 0.$$

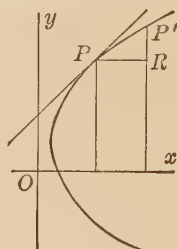


FIG. 70

The quantity $\frac{\Delta y}{\Delta x}$ occurring here is evidently the slope of the secant PP' . If now we let Δx approach 0, so that P' approaches P along the curve, Δy will also approach 0, but by § 75 the ratio $\frac{\Delta y}{\Delta x}$ will approach as its limit the slope m of the curve at P . Hence, when Δx approaches 0, (2) becomes

$$3m - 3 = 0, \quad \text{or} \quad m = 1.$$

(In the second term, $\frac{\Delta y}{\Delta x} \Delta y$, the factor $\frac{\Delta y}{\Delta x}$ approaches m while the factor Δy approaches 0, so that the whole term approaches 0.) Thus the equation of the tangent is

$$y - 2 = x - 1, \quad \text{or} \quad x - y + 1 = 0.$$

The method may now be summarized as follows:

To find the slope m of a given curve at a point $P: (x_1, y_1)$ on the curve, choose a neighboring point $P': (x_1 + \Delta x, y_1 + \Delta y)$ on the curve, substitute the coördinates of P' in the equation of the curve, and simplify. Divide through by Δx .

Let Δx approach 0, $\frac{\Delta y}{\Delta x}$ at the same time approaching the value* m , and solve for m .

Since the development of this process in no way depends on the degree of the equation, it follows that the method applies not only to the conics, but to curves in general.

77. Normal; subtangent; subnormal. A line perpendicular to the tangent at a point of a curve is called the *normal* to the curve at that point. The equation of the normal can evidently be written at once (§ 29) when that of the tangent is known.

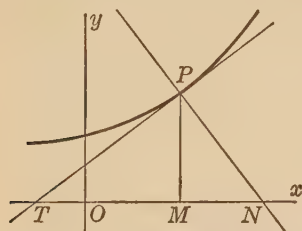


FIG. 71

If the tangent and normal at P cross the x -axis at T and N respectively, and if OM is the abscissa of P , then the distances TM and MN are called the *subtangent* and the *subnormal* corresponding to the point P .

EXERCISES

Find the tangent to the curve at the point indicated. In Exs. 1-6, draw the figure, and find the normal, subtangent, and subnormal.

1. $y^2 = x$ at $(4, 2)$. Ans. $x - 4y + 4 = 0$; $4x + y = 18$; 8 ; $\frac{1}{2}$.

2. $x^2 + y = 0$ at $(1, -1)$. Ans. $2x + y = 1$; $x - 2y = 3$; $\frac{1}{2}$; 2 .

3. $x^2 + 2y^2 = 3$ at $(1, -1)$. Ans. $x - 2y = 3$; $2x + y = 1$; 2 ; $\frac{1}{2}$.

* Exceptionally, $\frac{\Delta y}{\Delta x}$ approaches no limit. In the case of the conics, this means that the tangent is the line through P parallel to Oy —i.e. the line $x = x_1$.

4. $4x^2 + y^2 = 8$ at $(1, 2)$. *Ans.* $2x + y = 4$; $2y = x + 3$; 1; 4.
 5. $x^2 - y^2 = 9$ at $(5, 4)$. *Ans.* $5x - 4y = 9$; $4x + 5y = 40$; $\frac{16}{5}$; 5.
 6. $xy = 3$ at $x = -1$. *Ans.* $3x + y + 6 = 0$; $x - 3y = 8$; 1; 9.
 7. $x^2 + y^2 - x - 3y = 0$ at $(0, 0)$. *Ans.* $x + 3y = 0$.
 8. $(x + 1)^2 = 2(y - 3)$ at $x = 1$. *Ans.* $2x - y + 3 = 0$.
 9. $x^2 - 2xy + y^2 + 2x + y - 6 = 0$ at $(2, 2)$. *Ans.* $2x + y = 6$.
 10. $x^2 - xy + 3x - y = 1$ at $x = 2$. *Ans.* $4x - 3y + 1 = 0$.
 11. $y = x^3 - 2x - 1$ at $x = 2$. *Ans.* $y = 10x - 17$.
 12. $y = x^3 - 3x^2 + 2x - 5$ at $x = 1$. *Ans.* $x + y + 4 = 0$.
 13. $x^2 + 2y^2 - 2x + 4y + 2 = 0$ at $(2, -1)$. *Ans.* $x = 2$.

78. Tangent at a given point of a standard conic. Let $P : (x_1, y_1)$ be any point on the parabola

$$(1) \quad y^2 = 4ax,$$

and $P' : (x_1 + \Delta x, y_1 + \Delta y)$ a neighboring point on the curve. Substituting the coördinates of P' in (1), we get

$$(2) \quad y_1^2 + 2y_1\Delta y + \overline{\Delta y}^2 = 4ax_1 + 4a\Delta x.$$

Since (x_1, y_1) lies on the curve we have

$$(3) \quad y_1^2 = 4ax_1,$$

whence (2) becomes

$$2y_1\Delta y + \Delta y^2 = 4a\Delta x,$$

or

$$2y_1 \frac{\Delta y}{\Delta x} + \frac{\Delta y}{\Delta x} \Delta y = 4a.$$

When Δx approaches 0, this reduces to

$$2y_1 m = 4a, \quad m = \frac{2a}{y_1}.$$

The desired equation is therefore

$$y - y_1 = \frac{2a}{y_1}(x - x_1), \quad \text{or} \quad y_1 y - y_1^2 = 2ax - 2ax_1.$$

Simplifying by means of (3), we obtain the result:

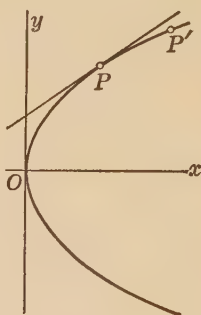


FIG. 72

The equation of the tangent to the parabola

$$y^2 = 4ax$$

at the point (x_1, y_1) on the curve is

$$(4) \quad y_1 y = 2ax + 2ax_1.$$

In a similar way we may prove that:

*The equation of the tangent to the central conic **

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

at the point (x_1, y_1) on the curve is

$$(5) \quad \frac{x_1 x}{a^2} \pm \frac{y_1 y}{b^2} = 1.$$

79. Subtangent and subnormal for the standard parabola. By (4), § 78, the x -intercept of the tangent to the standard parabola is

$$OT = -x_1;$$

hence the subtangent is

$$TM = 2x_1.$$

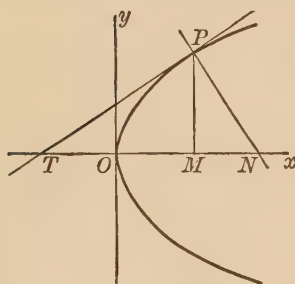


FIG. 73

That is, the subtangent for the parabola $y^2 = 4ax$ is *bisected at the vertex.*

The equation of the normal at P is (§ 29)

$$y_1 x + 2ay = x_1 y_1 + 2ay_1.$$

The x -intercept of this line is

$$ON = x_1 + 2a,$$

so that the subnormal is

$$MN = ON - OM = 2a.$$

That is, *the subnormal is constant*, and equal to the distance from the directrix to the focus.

* Since the ellipse and hyperbola have a center, they are called *central conics*.

EXERCISES

Find the tangent to the conic at the given point by the formulas of § 78.

1. $2y^2 + x = 0$ at $(-8, -2)$.
2. $3y^2 = 4x$ at $(3, -2)$.
3. $3x^2 + 4y^2 = 16$ at $(2, -1)$.
4. $x^2 + 6y^2 = 100$ at $(-2, 4)$.
5. $2x^2 - 3y^2 = 6$ at $(-3, 2)$.
6. $5x^2 - y^2 = 20$ at $(3, -5)$.

Find the tangent to the conic at the given point on the curve.

7. $x^2 = 4ay$ at (x_1, y_1) . *Ans.* $x_1x = 2ay + 2ay_1$.
8. $2xy = a^2$ at (x_1, y_1) . *Ans.* $y_1x + x_1y = a^2$.
9. $x^2 + y^2 = 2ax$ at (x_1, y_1) . *Ans.* $x_1x + y_1y = a(x + x_1)$.
10. $x^2 - y^2 = 2ay$ at (x_1, y_1) . *Ans.* $x_1x - y_1y = a(y + y_1)$.

11. Prove analytically that the tangents at the ends of the latus rectum of a parabola intersect on the directrix.

12. Prove that any tangent to a parabola intersects the directrix and the latus rectum (produced) in points equally distant from the focus.

13. Prove the theorem of Ex. 11 for the central conics.

14. Tangents are drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and to the circle $x^2 + y^2 = a^2$ at points having the same abscissa. Prove that these tangents cross Ox at the same point.

15. Prove that the perpendicular from a focus upon any tangent to an ellipse, and the line joining the center to the point of contact, intersect in a point on the directrix.

16. Prove the theorem of Ex. 15 for the hyperbola.

17. Prove that the product of the distances from the foci to any tangent to an ellipse is constant (equal to b^2).

18. Prove the theorem of Ex. 17 for the hyperbola.

Solve the following problems by ruler and compass.

19. Given a parabola and its vertex, draw the tangent at any point.

20. Given the axis, vertex, and one point of a parabola, construct the focus and directrix.

21. Given the axis and one point of a parabola, with the tangent at that point, construct the focus and directrix.

22. Given the axis, vertex, and one tangent to a parabola, construct the focus and directrix.

23. Given the tangent at the vertex of a parabola (vertex not marked), and another tangent with its point of contact, construct the focus.

24. Construct the tangent at any point of an ellipse. (Ex. 14.)

80. Quadratic discriminant. We know from algebra that the roots of the quadratic equation

$$ax^2 + bx + c = 0$$

are

real and different if $b^2 - 4ac > 0$;

real and equal if $b^2 - 4ac = 0$;

imaginary if $b^2 - 4ac < 0$;

and conversely. The quantity $b^2 - 4ac$ is called the *discriminant* of the quadratic.

81. The condition for tangency. It is obvious from the definition of tangent that when the equation of a curve and that of a line tangent to it are solved as simultaneous equations, two of the pairs of values of x and y will be the same: i.e. *two of the points of intersection will coincide.*

Let us now confine our attention to curves of the *second degree*. If the line and the curve are tangent, the quadratic in x (or y) obtained by substituting in the equation of the curve the value of y (or x) from the equation of the line will have equal roots. Conversely, *if this quadratic has equal roots the line is tangent to the curve*, since the points of intersection then have equal abscissas (or ordinates) and must coincide.* The condition

$$(1) \qquad b^2 - 4ac = 0$$

applied to the quadratic obtained in this way is called the *condition for tangency* of the line and the curve.† If either the equation of the line or that of the curve contains an undetermined constant, the condition for tangency serves to determine that constant.

* If the line is parallel to the y -axis the points have equal abscissas without necessarily coinciding. But in that case the equation of the line is $x = k$, so that in the process of solving the simultaneous equations we are led to a quadratic in y , and if this has equal roots the line is a tangent.

† This condition applies only to curves of the second degree, since it is only in this case that the process outlined leads, in general, to a quadratic equation in x or y . Tangents to higher curves are found by means of the differential calculus.

82. Tangents having a given slope. The theory of § 81 will now be applied to an

Example: Find the tangents to the circle $x^2 + y^2 = 4$ parallel to the line $y = x - 1$.

The equation of any line parallel to $y = x - 1$ is

$$(1) \quad y = x + k.$$

To determine k so that this line shall intersect the circle in two coincident points, substitute the value of y from (1) in the equation of the circle:

$$x^2 + (x + k)^2 = 4,$$

or

$$2x^2 + 2kx + k^2 - 4 = 0.$$

The roots of this quadratic in x are the abscissas of the points of intersection of the line (1) with the circle; if these roots are equal the points will coincide. We therefore set $b^2 - 4ac = 0$; i.e.

$$(2k)^2 - 4 \cdot 2(k^2 - 4) = 0, \quad -k^2 + 8 = 0,$$

whence

$$k = \pm 2\sqrt{2},$$

and the required tangents are *

$$y = x \pm 2\sqrt{2}.$$

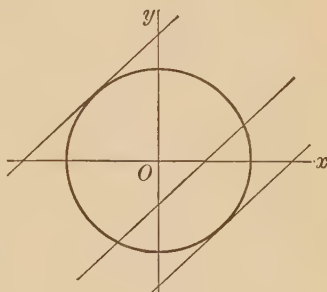


FIG. 74

* It may be objected that the above process does not rest squarely upon the definition of tangent, since the tangent here appears as the limiting position of a secant which moves parallel to its original position until the two points of intersection coincide. For the higher plane curves, this objection is valid (as an example, draw a line parallel to the y -axis in Fig. 94, p. 171, and let it approach the y -axis). However, in the case of the conic sections the same limiting position is obtained by this manner of approaching the limit as by the method used in the definition (§ 75). An analytic proof of this statement may be written out by the method of § 76.

EXERCISES

Find the equations of the following tangents.

1. To the circle $x^2 + y^2 = 13$ parallel to the line $3x - 2y = 6$.
Ans. $3x - 2y = \pm 13$.
2. To the circle $x^2 + y^2 = 5$ perpendicular to the line $x + 5y = 1$.
Ans. $5x - y = \pm \sqrt{130}$.
3. To the parabola $y^2 - 3x + 2y + 1 = 0$ perpendicular to the line $3x + 2y = 0$.
Ans. $16x - 24y + 3 = 0$.
4. To the parabola $3x^2 + x - y = 10$ parallel to the line $y = 13x$.
Ans. $y = 13x - 22$.
5. To the ellipse $x^2 + 4y^2 - 4x + 6y = 0$ parallel to the line $2x = 3y$.
6. To the ellipse $3x^2 + y^2 + x + 2y = 0$ perpendicular to the line $4x - 2y = 1$.
Ans. $x + 2y = 0$, $3x + 6y + 13 = 0$.
7. To the hyperbola $x^2 - 4y^2 - 3x + 6y + 6 = 0$ perpendicular to the line $y = 10x + 2$.
8. To the hyperbola $3x^2 - 2y^2 + x + 2y = 2$ parallel to the line $13x + 10y = 9$.
9. To the parabola $4x^2 - 12xy + 9y^2 + 3x - 2y = 0$ perpendicular to the line $2x + 3y + 5 = 0$.
Ans. $3x - 2y = 0$.
10. To the conic $2x^2 - 4xy + 2y^2 = x$ perpendicular to the line $4x + 5y = 0$.
Ans. $5x - 4y + 2 = 0$.
11. To the conic $3x^2 + xy + y^2 + x + 3y = 0$ parallel to the line $x + 3y = 2$.
Ans. $x + 3y = 0$, $11x + 33y + 100 = 0$.
12. To the conic $x^2 - 4xy + y^2 + 4x - 2y = 0$ parallel to the line $x - 2y = 3$.
13. To the ellipse $x^2 + 2xy + 2y^2 + 4x + 4y = 0$ parallel to the x -axis. Use the result to find the highest and lowest points of the curve, and the center.
Ans. $(-4, 2)$, $(0, -2)$; $C: (-2, 0)$.
14. Solve Ex. 13 for the ellipse $x^2 - 3xy + 4y^2 - 3x + 7y - 2 = 0$.
Ans. $(-\frac{1}{4}, -\frac{1}{4})$, $(3, 1)$; $C: (\frac{3}{4}, -\frac{5}{4})$.
15. Find the equations of the horizontal and vertical lines that box in the ellipse $x^2 + xy + y^2 + x = 0$.
Ans. $y = 1$, $y = -\frac{1}{3}$; $x = 0$, $x = -\frac{4}{3}$.
16. Solve Ex. 15 for the ellipse $x^2 - 2xy + 2y^2 + 2x - 4y + 1 = 0$.
Ans. $y = 0$, $y = 2$; $x = \pm \sqrt{2}$.
17. By two methods (§§ 82, 40), make the circle $x^2 + y^2 = 2ay$ touch the line $4x + 3y = 10$.
Ans. $x^2 + y^2 + 10y = 0$, $2x^2 + 2y^2 = 5y$.
18. By two methods, make the circle $x^2 + y^2 = 4ax - 2ay$ touch the line $y = 2x - 20$. Draw the figure.
Ans. $x^2 + y^2 = 8x - 4y$.

19. Make the parabola $y^2 = 4ax$ touch the line $2x + 3y = 4$.

$$\text{Ans. } 9y^2 + 32x = 0.$$

20. Make the hyperbola $x^2 - y^2 = a^2$ touch the line $2x + y = 3$.

$$\text{Ans. } x^2 - y^2 = 3.$$

21. Make the parabola $y = ax^2 + bx$ touch the line $x - 4y + 9 = 0$ at (3, 3).

$$\text{Ans. } x^2 - 7x + 4y = 0.$$

22. Make the parabola $y = ax^2 + bx$ touch the lines $x - y = 1$, $5x - y = 1$. (Write out the condition for tangency of each line to the curve; solve for a and b .)

$$\text{Ans. } y = x^2 + 3x.$$

23. Make the conic $Ax^2 + By^2 = 1$ touch the lines $x - 5y + 4 = 0$, $2x + 5y + 2 = 0$.

$$\text{Ans. } 5y^2 - x^2 = 4.$$

24. Find the common tangents to the curves $x^2 + y^2 = 2a^2$, $y^2 = 8ax$. (Write the condition for tangency of the line $y = mx + k$ to each conic; solve for m and k .)

$$\text{Ans. } y = \pm (x + 2a).$$

25. Find the common tangents to the curves $x^2 + y^2 + 10x = 20$, $y^2 = 20x$.

$$\text{Ans. } 2y = \pm (x + 20).$$

26. Find the common tangents to the parabolas $y^2 = 4x$, $x^2 = 2y - 3$. Draw the figure.

$$\text{Ans. } y = x + 1, 4x + 2y + 1 = 0.$$

83. Tangents of given slope to the standard conics. Using the quadratic discriminant, we may easily prove that:

For all values of m , except as noted below, the line

$$(1) \quad y = mx + \frac{a}{m}$$

is tangent to the parabola $y^2 = 4ax$;

the lines

$$(2) \quad y = mx \pm \sqrt{a^2 m^2 + b^2}$$

are tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;

the lines

$$(3) \quad y = mx \pm \sqrt{a^2 m^2 - b^2}$$

are tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

In (1) the value $m = 0$ is of course excluded. In (3), we must have $a^2m^2 - b^2 > 0$, i.e. $m > \frac{b}{a}$ or $m < -\frac{b}{a}$. For, if $m = \pm \frac{b}{a}$, the lines are the asymptotes (limiting positions of the tangent as the point of tangency recedes indefinitely); if $a^2m^2 - b^2 < 0$, i.e. $-\frac{b}{a} < m < \frac{b}{a}$, the constant term is imaginary.

84. Two important geometric properties of the parabola.

THEOREM I: *The tangent bisects the angle between the focal radius drawn to the point of contact and the line through that point parallel to the axis of the parabola.*

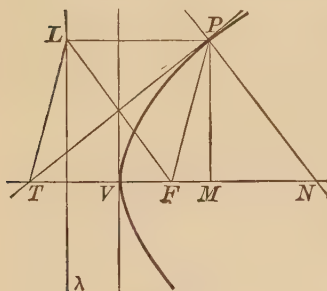


FIG. 75

THEOREM II: *The foot of the perpendicular from the focus upon any tangent lies on the tangent at the vertex.*

That is, in Fig. 75, the line PT bisects the angle FPL , and the line FL intersects PT on the tangent at V .

The proof of these theorems is left to the student.

EXERCISES

1. Prove formula (1), § 83.
2. Prove formulas (2), (3), § 83.
3. Prove that the tangents drawn to a parabola from any point of the directrix are perpendicular.
4. State and prove the converse of the theorem of Ex. 3.
5. Solve Ex. 12, p. 123, by a new method.
6. Solve Ex. 17, p. 123, by a new method.
7. Solve Ex. 18, p. 123, by a new method.
8. In Fig. 75, prove that $FP = FT = FN$.

9. Prove Theorem I of § 84 geometrically.
10. Prove Theorem II of § 84 geometrically.
11. Prove Theorem II analytically, using (1), § 83.
12. Prove Theorem II analytically, using (4), § 78.
13. Solve Ex. 19, p. 123, by a second method.
14. Solve Ex. 19, p. 123, by a third method.
15. Solve Ex. 21, p. 123, by a second method.
16. Solve Ex. 21, p. 123, by a third method.
17. Given the axis, the focus, and any tangent to a parabola, construct the directrix by two methods.
18. Given the focus of a parabola, and any tangent with its point of contact, construct the directrix.
19. Given the directrix and one point of a parabola, with the tangent at that point, find the focus.
20. Given the focus and two tangents to a parabola, find the vertex.

85. Tangents through an external point. Consider an

Example: Find the tangents to the circle

$$(1) \quad x^2 + y^2 = 20$$

through the point (6, 2). (See Fig. 42, p. 66.)

The equation of any line through (6, 2) is

$$y - 2 = m(x - 6),$$

or

$$(2) \quad y = mx - 6m + 2.$$

Substituting this value of y in (1), we get

$$x^2 + (mx - 6m + 2)^2 = 20,$$

which after expanding and collecting terms becomes

$$(1 + m^2)x^2 + 4m(-3m + 1)x + 36m^2 - 24m - 16 = 0.$$

This quadratic will have equal roots, and the line (2) will be a tangent, if $b^2 - 4ac = 0$: i.e. if

$$16m^2(-3m + 1)^2 - 4(1 + m^2)(36m^2 - 24m - 16) = 0.$$

This reduces to

$$2m^2 - 3m - 2 = 0,$$

whence

$$m = 2 \quad \text{or} \quad -\frac{1}{2},$$

and the tangents are

$$2x - y = 10, \quad x + 2y = 10.$$

Evidently this method enables us to find the tangents to any conic through an external point.

EXERCISES

1. Solve Ex. 29, p. 68, by a new method.
2. Solve Ex. 30, p. 68, by a new method.
3. Solve Ex. 33, p. 68, by a new method.
4. Solve Ex. 34, p. 68, by a new method.

Find the tangents to the conic through the given point.

5. To the parabola $y^2 + 3x = 0$ through $(-1, -2)$.
Ans. $x - 2y = 3, 3x - 2y = 1$.
6. To the parabola $x^2 = 6y + 10$ through $(7, 5)$.
Ans. $4x = 3y + 13, 3y = 10x - 55$.
7. To the ellipse $3x^2 + 4y^2 = 16$ through $(4, -2)$.
Ans. $3x + 2y = 8, y + 2 = 0$.
8. To the hyperbola $x^2 - 2y^2 = 7$ through $(-1, 5)$.
Ans. $3x + 2y = 7, 6y = 19x + 49$.
9. To the curve $xy = 2$ through $(6, -1)$.
Ans. $x + 18y + 12 = 0, x + 2y = 4$.
10. To the curve $x^2 - y^2 - 2x - 2y + 4 = 0$ through $(0, 1)$.
Ans. $y = 1, 4x + 5y = 5$.
11. To the hyperbola $x^2 - xy - y^2 + x + 2y = 0$ through $(-2, 1)$.
Ans. $x + 2y = 0, x - 3y + 5 = 0$.
12. To the parabola $2x^2 + 4xy + 2y^2 + x = 0$ through $(2, -2)$.
Ans. $5x + 4y = 2, 3x + 4y + 2 = 0$.
13. To the circle $x^2 + y^2 = 2x$ through $(0, 6)$. Draw the figure.
Ans. $35x + 12y = 72, x = 0$.
14. To the conic $(x - y)^2 = x + 1$ through $(1, 2)$.
Ans. $4y = 5x + 3, x = 1$.
15. Solve Ex. 5 by substituting the coördinates $(-1, -2)$ in (1), § 83.

16. Find tangents to the parabola $y^2 = 5x$ through $(3, -4)$ by two methods. (Cf. Ex. 15.)

17. Solve Ex. 7 by use of (2), § 83.

18. Solve Ex. 8 by use of (3), § 83.

19. Prove that, for all values of $m < 0$, the line $y = mx \pm a\sqrt{-2m}$ is tangent to the hyperbola $2xy = a^2$.

20. Use the formula of Ex. 19 to solve Ex. 9.

21. In applying the method of § 85, if the values of m turn out to be equal, what conclusion can be drawn? What if m is imaginary?

22. In applying the method of § 85, if the equation to determine m turns out to be of the first degree, what conclusion can be drawn? (Cf. Exs. 13, 14.)

23. Employ the method of § 85 in Ex. 13, p. 121.

24. Find the tangents to the conic $2x^2 + xy - 3y^2 - 10x - 3y + 12 = 0$ through $(3, 0)$.
Ans. $x = 3$.

86. Chord of contact. If tangents are drawn to a conic through an external point P , the line through the points of contact is called the *chord of contact* of P , with reference to the given conic. The term is also used as in elementary geometry, to mean the line segment joining the points of contact.

To find the chord of contact of the external point $P: (x_1, y_1)$ with respect to the parabola

$$y^2 = 4ax,$$

let QR be the required chord, and let the coördinates of Q and R (which are of course unknown) be denoted by (x_2, y_2) and (x_3, y_3) respectively. Then by (4), § 78, the equation of the tangent at Q is

$$y_2y = 2ax + 2ax_2,$$

that of the tangent at R is

$$y_3y = 2ax + 2ax_3.$$

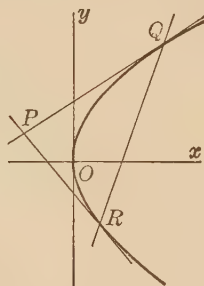


FIG. 76

By hypothesis, both these lines pass through P : substituting the coördinates of P for x and y in the two equations we obtain the relations

$$(1) \quad y_2 y_1 = 2ax_1 + 2ax_2,$$

$$(2) \quad y_3 y_1 = 2ax_1 + 2ax_3.$$

Consider now the straight line

$$(3) \quad y_1 y = 2ax + 2ax_1.$$

This line passes through Q , for if we substitute in (3) the coördinates (x_2, y_2) we obtain (1), which is known to be true. Similarly we see from (2) that the line (3) passes through R . This line is therefore the required chord of contact. Hence:

If the point $P : (x_1, y_1)$ is outside the parabola

$$y^2 = 4ax,$$

the equation

$$(4) \quad y_1 y = 2ax + 2ax_1$$

represents the chord of contact corresponding to P .

In the same way we find:

The equation of the chord of contact corresponding to the external point $P : (x_1, y_1)$, with reference to the conic

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

is

$$(5) \quad \frac{x_1 x}{a^2} \pm \frac{y_1 y}{b^2} = 1.$$

The proof is left to the student

EXERCISES

1. By employing the chord of contact, devise a new method for finding tangents to the standard conics through an external point. Employ this method to solve example (b), § 40.

Solve the following by the method suggested in Ex. 1.

2. Find tangents to the circle $x^2 + y^2 = 4$ through $(2, 1)$.

3. Ex. 5, p. 130.

4. Ex. 16, p. 131.

5. Ex. 7, p. 130.

6. Ex. 8, p. 130.

7. Prove formula (5), § 86.

8. Using the answer to Ex. 8, p. 123, find the equation of the chord of contact of an external point $P : (x_1, y_1)$ with reference to the hyperbola $2xy = a^2$.

Ans. $y_1x + x_1y = a^2$.

9. Using Ex. 9, p. 123, find the equation of the chord of contact of an external point $P : (x_1, y_1)$ with reference to the circle $x^2 + y^2 = 2ax$.

10. Solve Ex. 9, p. 130, by a new method (Ex. 8).

11. Solve Ex. 13, p. 130, by a new method (Ex. 9).

12. For the parabola, prove that the chord of contact of any point on the directrix passes through the focus.

13. State and prove the converse of the theorem of Ex. 12.

14. Prove the theorem of Ex. 12 for the central conics.

15. For the parabola, prove that the chord of contact of any point on the directrix is perpendicular to the line joining that point to the focus.

16. State and prove the converse of the theorem of Ex. 15.

17. Prove the theorem of Ex. 15 for the central conics.

18. For the parabola, prove that if a point lies on the latus rectum (produced), its chord of contact passes through the point of intersection of the axis and the directrix.

19. Prove the theorem of Ex. 18 for the central conics.

20. The points P_1, P_2 being external to a parabola, prove that if P_1 lies on the chord of contact of P_2 , then P_2 lies on the chord of P_1 .

21. Prove the theorem of Ex. 20 for the central conics.

22. Given a parabola with its axis, construct the tangents through a given external point. (Ex. 20 above, and Ex. 19, p. 123.)

23. Given an ellipse with its axes, construct the tangents through a given external point. (Ex. 21 above, and Ex. 24, p. 123.)

24. If O is the center of a circle, P an external point, and λ the chord of contact of P , prove that OP is perpendicular to λ .

25. In Ex. 24, if K is the point of intersection of OP with λ , prove that $OP \cdot OK = a^2$.

26. Prove that if the chord of contact of a point P with respect to the circle $x^2 + y^2 = a^2$ touches the circle $4x^2 + 4y^2 = a^2$, then P lies on the circle $x^2 + y^2 = 4a^2$.

II. DIAMETERS

87. Diameter of a conic. Every diameter of a circle (as defined in § 36) bisects a certain set of parallel chords, and no other line does so; hence this property might equally well have been employed as a definition of "diameter."

In introducing the term with reference to the conics, we will adopt the definition thus suggested:

A diameter of a conic section is the locus of the midpoints of a system of parallel chords.

To the loci thus defined it will be convenient to add, by agreement, the asymptotes of a hyperbola.

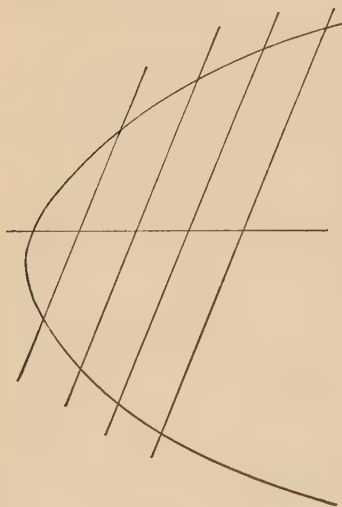


FIG. 77

Though not obvious from the definition, it will appear in § 89 that a diameter as thus defined is either a straight line or a portion of a straight line. In the latter case the term will also be used to denote the entire straight line.* Further, it will appear that for the central conics the definition of diameter as a straight line through the center would be equivalent to the definition just given.

Before proceeding further, it will be necessary to recall the following theorem of algebra:

In the quadratic equation $ax^2 + bx + c = 0$, the sum of the roots is $-\frac{b}{a}$.

* A similar dual definition was laid down in § 36.

88. Diameter bisecting a given set of parallel chords.
The method of finding the equation of the diameter bisecting a given system of parallel chords will be explained by an

Example: Find the diameter of the ellipse

$$(1) \quad 2x^2 + 3y^2 = 6$$

bisecting the chords

$$(2) \quad x + y = k.$$

Let $P_1 : (x_1, y_1)$ and $P_2 : (x_2, y_2)$ be the endpoints, and $P : (x, y)$ the midpoint, of any one of the given chords, so that P is a general point on the re-

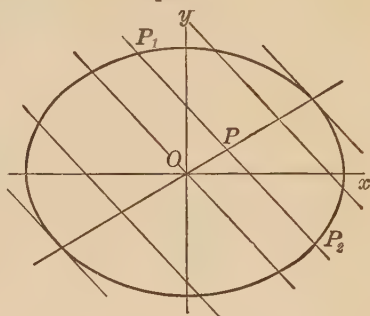


FIG. 78

quired locus. We now have to obtain an equation involving x and y but not involving x_1, y_1, x_2, y_2 , or k . Substituting the value of y from (2) in (1), we get

$$5x^2 - 6kx + 3k^2 - 6 = 0.$$

Now the roots of this quadratic in x are evidently the abscissas of the endpoints P_1, P_2 . By the theorem of § 87,

$$x_1 + x_2 = \frac{6}{5}k.$$

But since P is the midpoint of P_1P_2 , its abscissa is (§ 7)

$$x = \frac{1}{2}(x_1 + x_2),$$

or

$$(3) \quad x = \frac{3}{5}k.$$

Substituting the value of k from (2) in (3), we find the required diameter to be

$$x = \frac{3}{5}(x + y), \text{ or } 2x - 3y = 0.$$

EXERCISES

Find the equations of the following diameters.

1. Of the ellipse $x^2 + 3y^2 = 3$ bisecting the chords $y = 3x + k$.
Ans. $x + 9y = 0$.
2. Of the ellipse $5x^2 + 2y^2 = 10$ bisecting the chords $2x + 3y = k$.
Ans. $4y = 15x$.
3. Of the hyperbola $x^2 - 3y^2 = 6$ bisecting chords of slope $\frac{3}{2}$.
Ans. $2x = 9y$.
4. Of the hyperbola $4x^2 - 4y^2 = 3$ bisecting chords parallel to the line $x + 4y = 2$.
Ans. $4x + y = 0$.
5. Of the parabola $y^2 = 4x + 2y$ bisecting chords perpendicular to the line $x + 2y = 0$.
Ans. $y = 2$.
6. Of the parabola $x^2 - x + 4y + 1 = 0$ bisecting chords perpendicular to the line $4x - 5y = 3$.
Ans. $x = 3$.
7. Of the ellipse $x^2 + 2y^2 + x + y = 0$ bisecting chords parallel to the line $x - 3y = 2$.
Ans. $3x + 2y + 2 = 0$.
8. Of the hyperbola $3x^2 - y^2 = 2x + 4y$ bisecting the chords $2x + 3y = k$.
Ans. $9x + 2y + 1 = 0$.
9. Of the hyperbola $3x^2 - xy = 3$ bisecting chords of slope -2 .
10. Of the hyperbola $2xy + y^2 = 4$ bisecting chords of slope $\frac{3}{2}$.
11. Bisecting the chords of slope $-\frac{1}{2}$ of the parabola $3x^2 - 6xy + 3y^2 + 4x = 0$.
Ans. $9y = 9x + 4$.
12. Of the conic $4x^2 - 4xy + y^2 + 2x - 2y = 0$ bisecting chords of slope 3.
Ans. $y = 2x + 2$.
13. Bisecting those chords of the parabola $(x - y)^2 = 3x$ that are parallel to Ox .
Ans. $2x - 2y = 3$.
14. Bisecting those chords of the hyperbola $2x^2 - xy - 2y^2 + x - y + 3 = 0$ that are parallel to Oy .
Ans. $x + 4y + 1 = 0$.
15. Of the conic $3x^2 - 4xy + 2y^2 - x - 3y + 6 = 0$ bisecting chords parallel to the y -axis.
Ans. $4x - 4y + 3 = 0$.
16. Bisecting chords of the parabola $(2x + y)^2 + x - 2y = 0$ parallel to the line $3x - y = 4$.
17. In Ex. 7, show that the diameter passes through the center of the curve.
18. In Ex. 8, show that the diameter passes through the center of the curve.
19. In Ex. 11, show that the diameter is parallel to the axis of the parabola. (Cf. § 69.)

89. Diameters of the standard conics. By the method of § 88 we may easily obtain the following results:

Given the system of parallel chords

$$y = mx + k,$$

where m is given and k is arbitrary, the equation of the diameter bisecting these chords is, for the parabola $y^2 = 4ax$,

$$(1) \quad y = \frac{2a}{m}; \quad (m \neq 0)$$

for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

$$(2) \quad y = -\frac{b^2}{a^2m} x; \quad (m \neq 0)$$

for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

$$(3) \quad y = \frac{b^2}{a^2m} x. \quad (m \neq 0)$$

For the parabola, the case $m = 0$ is automatically excluded, since lines parallel to the axis intersect the curve in only one point (Ex. 20, p. 113). When $m = 0$ equations (2), (3) cannot be written in the form given; but, clearing of fractions and then setting $m = 0$, we obtain the conjugate axis as bisector of chords parallel to the transverse axis, which is obviously correct.

Since equation (1) represents a straight line parallel to Ox , while (2) and (3) are straight lines through the origin, we deduce at once

THEOREM I: *Every diameter of a parabola is a straight line parallel to the axis of the curve.*

THEOREM II: *Every diameter of a central conic is a straight line through the center.*

The proofs, however, are not quite complete. (Ex. 3 below.)

The converse theorems are also true; their proof is left to the student.

90. Geometric properties involving diameters. Two important properties of the conics are as follows:

THEOREM I: *The tangents at the ends of a diameter of a central conic are (the tangent at the end of a diameter of a parabola is) parallel to the bisected chords.*

THEOREM II: *The tangents at the ends of any chord* intersect on the diameter bisecting that chord.*

We will prove Theorem II for the ellipse. Take the equation of the curve as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and let the given chord be the line

$$(1) \quad \frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1.$$

Then, by § 86, the tangents at the ends of this chord intersect at the point (x_1, y_1) . Now the slope of the chord (1) is evidently

$$m = -\frac{b^2 x_1}{a^2 y_1}.$$

Substituting this value of m in (2), § 89, we find the equation of the corresponding diameter to be

$$y = \frac{y_1}{x_1} x.$$

Since this line passes through (x_1, y_1) , the theorem is proved.

* Excepting, of course, all diameters of a circle, and all chords passing through the center of a central conic, since in those cases the tangents at the ends do not intersect.

EXERCISES

1. Derive formula (1), § 89.
2. Derive (2) or (3), § 89.
3. The theorems of § 89 are incompletely proved, in that the chords $x = k$ have not been considered. Carry out the proof of each theorem for this case. What lines are obtained as diameters?
4. Prove that in general a diameter of a parabola is not perpendicular to its chords. What exception is there?
5. Solve Ex. 4 for the central conics. What exceptions are there?
6. Find the equation of the diameter bisecting chords of slope m of the parabola $x^2 = 4ay$. *Ans. $x = 2am$.*
7. Find the equation of the diameter bisecting chords of slope m of the rectangular hyperbola $2xy = a^2$. *Ans. $y + mx = 0$.*
8. Using the answer to Ex. 7, prove that the asymptotes of a rectangular hyperbola bisect the angles between any diameter and the corresponding chords.
9. Prove the theorem of Ex. 8, using (3), § 89.
10. Prove that the tangent at the end of any diameter of a parabola crosses the tangent at the vertex at a point midway between the vertex and the diameter.
11. Prove the converse of Theorem I, § 89.
12. Prove the converse of Theorem II, § 89.
13. Evaluate (3), § 89, for $m = \pm \frac{b}{a}$. Hence show that an asymptote may be considered as a diameter bisecting chords parallel to itself.
14. Prove that the tangent at the end of a diameter of a parabola is parallel to the bisected chords. (Theorem I, § 90.)
15. Prove Theorem I, § 90, for the central conics.
16. Prove Theorem II, § 90, for the parabola.
17. Given a parabola, find the vertex by ruler and compass.
18. Given a central conic, construct its center.
19. Draw the tangent to a parabola parallel to a given line.
20. Draw the tangents to an ellipse parallel to a given line.
21. Given a parabola and a point inside, draw the chord that is bisected at that point.
22. Given an ellipse and a point inside, draw the chord that is bisected at that point.

CHAPTER X

ALGEBRAIC CURVES

91. Algebraic curves. An *algebraic equation* in two variables x and y is an equation which can be written in the form of a simple polynomial in x and y , equated to 0. Thus equation (1), § 69, is the most general algebraic equation of the second degree; every equation of the third degree may be written in the form

$$(1) \quad a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3 + b_0x^2 + b_1xy + b_2y^2 + c_0x + c_1y + d_0 = 0;$$

etc.

An *algebraic curve* is a curve that can be represented by an algebraic equation in the cartesian coördinates x, y . All others are *transcendental curves*: e.g.

$$y = \sin x, \quad y = \log_{10} x, \quad y = 2^x; \text{ etc.}$$

All plane curves other than the straight line (curve of first degree) and the conic sections (curves of second degree) are called *higher plane curves*. Algebraic curves of the third degree are called *cubic curves*, or simply cubics; those of the fourth degree, *quartics*; etc. In the present chapter we will consider some of the simpler types of higher algebraic curves, chiefly cubics and quartics. Factorable equations (§ 18) and equations having no locus will be excluded.

As we know, every equation of the second degree can be reduced to a "standard form," after which the curve is readily traced. On account of the great variety of possible forms, no similar process is feasible even for the

cubics—much less so for curves of still higher degree. As remarked in § 13, the elementary method of point-plotting is of very limited usefulness. The only remaining alternative is to discover, in each case, as many as possible of the algebraic properties and peculiarities of the equation, and then to translate these into geometric language. (A beginning in this direction was made in §§ 14–16.) While the line of attack depends more or less on the particular equation in hand, nevertheless the process can be systematized to some extent, as will now be shown.

92. Asymptotes. If the tangent to a curve approaches a definite limiting position as its point of contact recedes to infinity, the line so approached is called an *asymptote*. Thus an asymptote is sometimes spoken of as “a tangent whose point of contact lies at infinity.”

Inspection of a figure strongly suggests that the asymptotes of a hyperbola as previously defined (§ 57) also satisfy the above definition, and this fact is easily established. While the hyperbola is the only conic having asymptotes, many higher plane curves possess one or more of these lines, and they play an important part in the geometry of those curves.

It can be shown that *an asymptote to a curve of the n -th degree may intersect the curve in, at most, $n-2$ points*. While the proof of this theorem will be omitted, it can be made plausible as follows. The last sentence of § 20 shows that a straight line may intersect a curve of the n -th degree in not more than n points. Since a tangent is the limiting position of a secant when two points of intersection come to coincidence in the manner described in § 75, it follows that the point of tangency counts as two intersections, and a tangent may intersect the curve in no more than

$n-2$ other points. While an asymptote is not a tangent in the ordinary sense, it is the limiting position of a tangent and partakes of the nature of a tangent; thus it is reasonable to expect that the same result will hold.*

93. Behavior of y for large values of x ; horizontal asymptotes. When x increases indefinitely in either direction, y of course may behave in a variety of ways. The two most frequently occurring situations are:

(a) y increases without limit, positively or negatively; or

(b) y approaches a definite limit a .

These results are easily interpreted. For example, in (a), if y becomes large and positive as x becomes large and positive, we know that the curve recedes indefinitely from both axes, in the first quadrant.

In (b), the curve approaches more and more closely the line $y = a$: this line is a *horizontal asymptote*.†

94. Vertical asymptotes. It may happen that y increases indefinitely as x approaches some value b . This means that the curve approaches more and more closely the line $x = b$: this line is in general a *vertical asymptote*.

95. Restriction to definite regions. It is usually possible to determine certain definite portions of the plane within which the curve must lie. While no general directions can be given, in many cases we can *solve the equation for y (or y^2)* and note the *changes of sign* of the right member. The process will be explained presently by examples.

When this step can be effectively carried out, it is one of the most helpful of all in tracing the curve.

* This checks with the fact that an asymptote to a hyperbola cannot intersect the curve, since, for the hyperbola, $n = 2$.

† In rare instances a curve approaches a line more and more closely but the tangent approaches no limiting position, so that the line is not an asymptote. Such exceptions cannot occur among algebraic curves.

96. Summary. The preceding remarks may now be brought together in the form of a definite sequence of steps, as follows:

1. *Test the curve for symmetry.* (§ 16).
2. *Find the intersections with the axes.* (§ 14).
3. *Determine the behavior of y for large values of x . Find the horizontal asymptotes.* (§ 93.)
4. *Find the vertical asymptotes.* (§ 94.)
5. *Determine as narrowly as possible those regions of the plane in which the curve lies.* (§ 95.)
6. *Look for any further general information that may be obtainable; and if necessary, plot a few points.**

97. Polynomials. We take first the case in which y is a *polynomial* in x , also called a *rational integral function*:

$$y = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

where n is a positive integer and $a_0, a_1, \cdots, a_{n-1}, a_n$ are rational numbers. The cases $n = 1, n = 2$ have already been studied (§§ 30, 51); thus we may now assume $n \geq 3$.

Before considering special examples, it will be well to apply our analysis step by step to the polynomial in general, thus deducing certain results applicable to all curves of this class.

1. The curve cannot be symmetric about the x -axis.
2. The x -intercepts are the real roots of the polynomial.
3. As x becomes large in either direction, y becomes large (though not necessarily of the same sign as x): at the extreme right and extreme left the curve recedes farther and farther from the x -axis.
4. This step may be omitted, since no polynomial curve can have an asymptote, vertical or otherwise.

* We are trying to obtain, not an accurate detailed drawing of the curve, but merely its general form; thus in most cases only a very limited amount of point-plotting is either necessary or desirable.

5. A polynomial can change sign only by passing through the value 0, hence the curve can cross the x -axis only by intersecting * it. The function actually will change sign at every x -intercept (x equal to a root) except when the vanishing factor carries an *even exponent* (double root, quadruple root, etc.), in which case the curve will touch Ox and turn back. See Figs. 79, 80.

6. General remarks: (a) For every value of x there is one and only one value of y . Hence the curve extends across the plane in an unbroken arc from left to right. (b) A curve of the n -th degree can intersect a straight line in only n points. Hence (and this remark applies to all non-degenerate algebraic curves) no segment of the curve is straight, and no part should ever be put in with a ruler. For if the curve contained a segment of a straight line, it would have in common with that line an infinite number of points.

Example: Trace the curve

$$y = (x - 1)^2(x^2 - 4).$$

1. There is no symmetry.
2. Intercepts: $(0, -4)$; $(1, 0)$, $(\pm 2, 0)$.
3. When x is large positive, y is large positive; x large negative, y large positive. The curve rises indefinitely in the first and second quadrants.
5. y changes sign as x passes through the values $-2, 2$. By step 3, at the extreme left $y > 0$; it therefore remains positive in the interval $x < -2$, becomes negative in the interval $-2 < x < 2$, positive for $x > 2$. The curve is restricted to the unshaded regions of the plane † in Fig. 79.

* A "discontinuous" curve may cross by jumping. See Fig. 68, p. 117, or Fig. 80, p. 148.

† In the actual drawing of curves it is not desirable to shade the figure in this way. The device is adopted here to show graphically the meaning of step 5.

Plotting the additional point * $(-1, -12)$, we draw the curve, with the x -scale four times the y -scale.

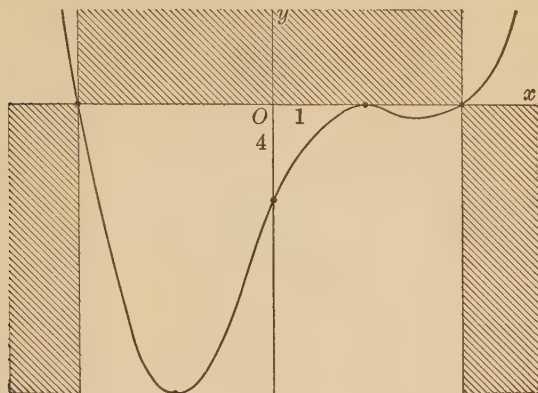


FIG. 79

EXERCISES

Trace the following curves, each on a suitable scale.

1. $y = x(x^2 - 1)$.
2. $y = x(4 - x^2)$.
3. $y = -x(x^2 - 6x + 5)$.
4. $y = x(x^2 + 5x + 6)$.
5. $y = (1 - x)^2(2 - x)$.
6. $y = x(x - 1)^2$.
7. $y = x^3 - 14x^2 + 59x - 70$.
8. $y = x^3 + 10x^2 + 29x + 20$.
9. $y = (x + 4)^3$.
10. $y = (1 - x)^3$.
11. $y = x^2(x^2 - 1)$.
12. $y = 1 - x^4$.
13. $y = 3x + x^2 - 3x^3 - x^4$.
14. $y = 4x^4 - 17x^2 + 4$.
15. $y = x(x - 2)^3$.
16. $y = 4x^3 - 3x^4$.
17. $y = x^4 - x^2 - 12$.
18. $y = x^4 - 3x^3 + 2x^2 - 6x$.
19. $y = (x - 1)^2(x^3 - 2x^2)$.
20. $y = x(x^2 - 1)(x - 2)^2$.

Draw the graphs of the following cubic functions, indicating the portion of the curve that has a meaning.

21. The volume of a cube of edge l . In the interval $0 < l < 1$, plot by points at l -intervals of 0.2.

* To find the *lowest* point on the curve, the differential calculus is required; it must not be supposed that $(-1, -12)$ is that point.

22. The volume remaining when a cube of edge x is removed from a cube of edge 2.

23. The volume remaining when a slab of thickness 1 is cut from one face of a cube of edge l .

24. The volume remaining when slabs of thickness x , $2x$, $3x$ are cut from three mutually perpendicular faces of a cube of edge 1.

25. The volume of a box made by cutting squares of side x out of the corners of a piece of cardboard 6 in. square and turning up the sides.

26. Ex. 25 if the cardboard is 8×4 inches.

27. The volume of a right circular cylinder of radius x inscribed in a right circular cone of radius 4 and height 8.

28. The volume of a right circular cylinder of altitude $2y$ inscribed in a sphere of radius 1.

29. The volume of a right circular cone of altitude y inscribed in a sphere of radius 1.

30. $\cos 3\theta$ as a function of $\cos \theta$: $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$. (Put $x = \cos \theta$, $y = \cos 3\theta$.)

31. $\sin 3\theta$ as a function of $\sin \theta$: $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$. ($x = \sin \theta$, $y = \sin 3\theta$.)

32. Show that the curve of Ex. 31 is the reflection, in the y -axis (§ 68), of the curve of Ex. 30.

98. Rational fractions. Consider next the case in which y is equal to a *rational fraction*—i.e. the quotient of two polynomials:

$$(1) \quad y = \frac{N}{D},$$

where

$$N = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, \quad (n \geq 0)$$

$$D = b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m. \quad (m \geq 1)$$

It will be assumed in this and the next article that our fraction is in its lowest terms—i.e. that N and D contain no common factor.

We first analyze the general problem.

1. No symmetry with respect to Ox .

2. The x -intercepts of course are the roots of N .

3. As x increases in either direction:

(a) If N is of higher degree than D ($n > m$), y becomes large, though not necessarily of the same sign as x . (A method of proof is suggested below.)

(b) If N is of lower degree than D , y approaches 0: the x -axis is an asymptote.

(c) If N and D are of the same degree, y approaches $\frac{a_0}{b_0}$: the line $y = \frac{a_0}{b_0}$ is an asymptote.

4. y becomes infinite when D becomes 0. Thus we find the real roots (if any) of the denominator, say $r_1, r_2, \dots$; the lines $x = r_1, x = r_2, \dots$ are vertical asymptotes.

5. A fraction changes sign when either N or D does so. Thus we list the roots of N and D (already found in steps 2 and 4), *casting out the roots of even order* (double, quadruple), and note for each of the others a change of sign of y and a passage of the curve across the x -axis: by intersection where $N = 0$, by jumping where $D = 0$.

6. There is one and only one value of y for every value of x , excepting those values for which $D = 0$.

Proof of 3(c): In this case, $m = n$:

$$y = \frac{a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n}{b_0x^n + b_1x^{n-1} + \dots + b_{n-1}x + b_n}.$$

Divide numerator and denominator by x^n :

$$y = \frac{a_0 + \frac{a_1}{x} + \dots + \frac{a_{n-1}}{x^{n-1}} + \frac{a_n}{x^n}}{b_0 + \frac{b_1}{x} + \dots + \frac{b_{n-1}}{x^{n-1}} + \frac{b_n}{x^n}}.$$

Then obviously, when x increases indefinitely, y approaches the limit $\frac{a_0}{b_0}$. Cases 3(a) and 3(b) may be handled in similar fashion.

Example: Trace the cubic curve

$$(2) \quad y = \frac{x^2}{(x-1)(x-3)}.$$

1. There is no symmetry.
2. The only intersection with the axes is $(0, 0)$.
3. By the above argument, as $x \rightarrow \pm \infty$, $y \rightarrow 1$: the line $y = 1$ is an asymptote. For large positive x , the denominator is less than the numerator and $y > 1$; for large negative x , $y < 1$. Thus the curve approaches the asymptote from above at the extreme right and from below at the extreme left. Finally, putting $y = 1$ in (2), we find $x = \frac{3}{4}$: the curve crosses the asymptote at $(\frac{3}{4}, 1)$.
4. As $x \rightarrow 1$, and as $x \rightarrow 3$, y increases indefinitely: the lines $x = 1$, $x = 3$ are asymptotes.
5. The numerator vanishes at $x = 0$, but does not change sign because of the even exponent; the denominator, and hence the function, changes sign as x goes through 1, 3. This combined with step 3 limits the curve as shown.

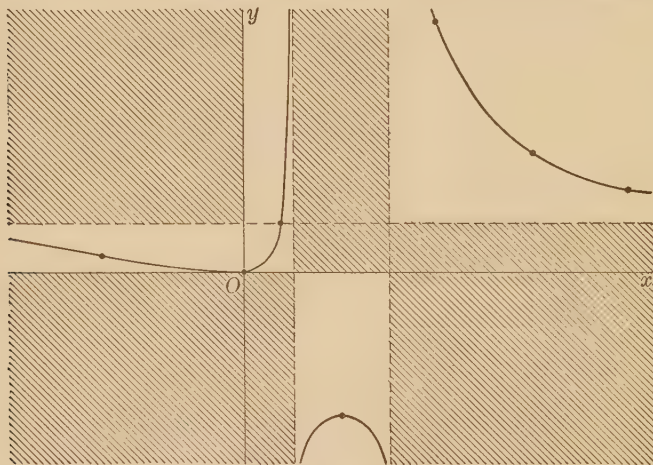


FIG. 80

EXERCISES

1. $y = \frac{2x}{1+x^2}$.

2. $y = \frac{1}{1+x^2}$.

3. $x^2y = 4y - 1$.

4. $x^2y = y - x$.

5. $x^3 - xy - 4x + y = 0$.

6. $x^3 - 4x^2 + xy + 2y = 0$.

7. $x^2y + x^2 - y - 4 = 0$.

8. $4x^2y - x^2 - 9y + 1 = 0$.

9. $2x^2y - x^2 + 3xy + 2x + y = 1$.

10. $x^2y - 3x^2 - y + 1 = 0$.

11. $y = \frac{1-2x+x^2}{x^2}$.

12. $y = \frac{1+2x-x^2-2x^3}{(x-2)^2}$.

13. $y = \frac{4x^3 - 16x^2 - x + 4}{(x-3)^2}$.

14. $y = \frac{3x^2 + 4x + 1}{x^2}$.

15. $y = \frac{(x-1)(x-2)^2}{x(x+1)}$.

16. $y = \frac{x^3 - 8x^2 + 19x - 12}{4x^2 - 9}$.

17. $y = \frac{1-4x^2}{x(x^2-1)}$.

18. $y = \frac{x^3 + x}{(x-1)(x^2-4)}$.

19. $y = \frac{4x^3 - 13x^2 + 6x + 3}{x^2(x-4)}$.

20. $y = \frac{(x^2-1)^2}{x^2(x+4)}$.

21. $y = \frac{(x^2-2)^2}{x^2(x-3)^2}$.

22. $y = \frac{2x^3 - 10}{x^3 - 3x^2 + 2x}$.

23. A right circular cone is circumscribed about a sphere of radius a . Express the volume of the cone as a function of its altitude, and draw the graph. What kind of curve is this?

$$\text{Ans. } V = \frac{1}{3}\pi a^2 \cdot \frac{h^2}{h-2a}.$$

24. In Ex. 23, express the volume of the cone as a function of its radius, and draw the graph.

$$\text{Ans. } V = \frac{2}{3}\pi a \cdot \frac{r^3}{r^2 - a^2}.$$

25. When does (1), § 98, represent a hyperbola? (Ex. 18, p. 113.) When an equilateral hyperbola? (§ 74.)

26. Draw the curve from which $\tan 2\theta$ may be read off when $\tan \theta$ is given. (Put $x = \tan \theta$, $y = \tan 2\theta$.)

$$27. \text{ Represent } \tan 3\theta \text{ as a function of } \tan \theta: \tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}.$$

(Put $x = \tan \theta$, $y = \tan 3\theta$.)

28. Draw a curve from which $\sin 2\theta$ may be read off if $\tan \theta$ is given: $\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$. (Cf. Ex. 1.)

29. Draw a curve from which $\cos 2\theta$ may be read off if $\tan \theta$ is given.

99. Two-valued functions. If to each value of x there corresponds one and only one value of y , then y is said to be a *one-valued* function. The polynomial, the rational fraction, and many other important functions possess this property.

If to each value of x there correspond two values of y , the function is *two-valued*. Two important classes of functions of this type are obtained by replacing y by y^2 in §§ 97–98.

$$(1) \quad y^2 = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$$

If $n = 1$, the curve is a parabola with axis in Ox ; if $n = 2$, a central conic with one axis in Ox . We consider now the case $n \geq 3$.

$$(2) \quad y^2 = \frac{N}{D},$$

where N and D are polynomials in x .

Instead of a general analysis, consider two

Examples: (a) Trace the curve

$$y^2 = (x - 2)(x - 1)(x + 1).$$

1. All curves of the type (1) are symmetric to Ox .
2. $(2, 0)$, $(1, 0)$, $(-1, 0)$; $(0, \pm \sqrt{2})$.
3. When x is large positive, y^2 is large positive and y is large positive and negative; when x is large negative, y^2 is negative and y imaginary.
4. Evidently no vertical asymptotes for the curves (1).
5. y^2 changes sign at $x = -1, 1, 2$. When $y^2 > 0$, y is real positive and negative; when $y^2 < 0$, y is imaginary. Since $y^2 < 0$ at the extreme left, there will be no curve until the first change of sign is reached, at $x = -1$; the curve appears in the interval $-1 < x < 1$, disappears for $1 < x < 2$, reappears for $x > 2$.

The oval of course is not an ellipse, and in fact is not symmetric to Oy , the highest point being slightly to the left (Fig. 81).

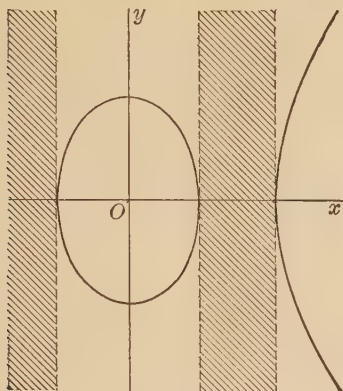


FIG. 81

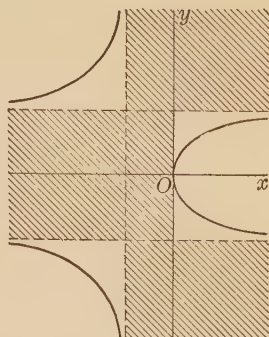


FIG. 82

(b) Trace the curve

$$(3) \quad y^2 = \frac{2x}{x+1}.$$

1. All curves of the type (2) are symmetric to Ox .
2. $(0, 0)$.
3. When x becomes large positive or negative, $y^2 \rightarrow 2$: the lines $y = \pm \sqrt{2}$ are asymptotes. For positive x , $y^2 < 2$ and the curve lies between the asymptotes; for negative x , $y^2 > 2$ and the curve lies outside of the strip bounded by those lines. Putting $y = \pm \sqrt{2}$ in (3), we see that there is no intersection with the asymptotes.
4. $x = -1$ is an asymptote.
5. y^2 changes sign as x goes through the values $-1, 0$. At the extreme left $y^2 > 0$: the curve is present in the interval $x < -1$, absent for $-1 < x < 0$, present for $x > 0$. This combined with step 3 limits the curve as shown by the shading (Fig. 82).

EXERCISES

Trace the following curves.

1. $y^2 = x(x-2)(x-3)$.

2. $y^2 = (x^2-4)(5-x)$.

3. $y^2 = x^4 - 5x^3 + 5x^2 - 5x + 4$.

4. $y^2 = x^4 - x^3 - 2x - 4$.

5. $y^2 = (4x^2-1)(1-x^2)$.

6. $y^2 = x(x-2)(1-x^2)$.

7. $y^2 = x - x^2 + 2x^3 - x^4$.

8. $y^2 = (2x^2+x)(x^2+2x+3)$.

9. $y^2 = \frac{1}{x^2+1}$.

10. $y^2 = \frac{2x}{x^2+1}$.

11. $x^2y^2 - x^2 + y^2 + 2x = 0$.

12. $4x^2y^2 - x^2 + y^2 + x + 2 = 0$.

13. $y^2 = \frac{(x+1)^2}{x^3}$.

14. $y^2 = \frac{(x-2)^2}{1-x^3}$.

15. $y^2 = \frac{x(x+2)}{x+1}$.

16. $y^2 = \frac{x^2+3x+2}{x-3}$.

17. $x^3 - xy^2 + 3y^2 - 7x + 6 = 0$.

18. $x^3 - xy^2 - y^2 - 7x + 6 = 0$.

19. $y^2 = \frac{x-1}{(x+1)^2}$.

20. $y^2 = \frac{x^3 - x^2 + 3x - 3}{x^2}$.

21. Show that if $\tan \theta$ is given, the values of $\cos \theta$ may be read off from the curve of Ex. 9. (Put $x = \tan \theta$, $y = \cos \theta$.)

22. Express $\sec \theta$ as a function of $\cos 2\theta$, and draw the graph.

23. If the total surface (including the base) of a right circular cone is given as 2π square units, express the altitude as a function of the radius, and draw the graph.

24. A right circular cone is circumscribed about a sphere of radius a . Express the radius of the cone as a function of its altitude, and draw the graph.

CHAPTER XI

POLAR COÖRDINATES

100. Distance and bearing. Instead of locating a point by its distances from two perpendicular lines (§ 2), we frequently, in ordinary usage, locate it by its distance and “bearing” from some fixed point: one town is 5 miles southeast of another; one boundary marker is 90 ft. N. 10° E. of another; etc. This alternative method, like the former one, has its counterpart in analytic geometry.

101. Polar coördinates. Let us choose a fixed line Ox in the coördinate plane, and a point O on this line. The position of any point P in the plane is determined if we know the length of the line OP together with the angle that this line makes with the fixed line Ox , both the distance and the angle being measured in a definite sense. The segment OP and the angle xOP are the *polar coördinates* of P ; they are called the *radius vector* and the *polar angle* respectively, and are denoted by the letters r, θ . The fixed line Ox is the *initial line*, or *polar axis*, and the point O is the *pole*, or *origin*.

The polar coördinates of a point are written in parentheses with the radius vector first, as $P : (r, \theta)$, or simply (r, θ) . The polar angle is *positive* when measured *counterclockwise*, *negative clockwise*; the radius vector is *positive* if laid off *on the terminal side of θ* , *negative* if measured in the opposite sense, i.e. *on the terminal side produced through O* . Fig. 83 shows the points $Q : (1, 30^\circ)$, $R : (-2, 30^\circ)$, $S : (-2, -\frac{1}{4}\pi)$.

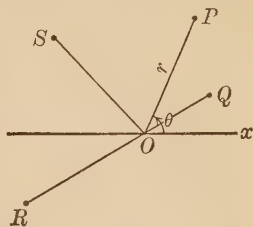


FIG. 83

To plot a point whose polar coördinates are given, it is best to begin by drawing the line on which the radius vector lies, i.e. the line making an angle θ with Ox , and then to lay off on that line, in the proper sense, the distance r .

102. The locus of an equation. If r and θ are connected by an equation of any form, we may assign values to θ and compute the corresponding value or values of r . The points thus determined all lie on a definite curve, the *locus* of the equation. Curve plotting is done most conveniently on "polar coördinate paper," which is paper ruled in concentric circles and radial lines.

While to every pair of polar coördinates corresponds a single definite point, the converse is not true: the same point may be represented by various pairs of coördinates. For instance, in Fig. 83, the coördinates $(1, 30^\circ)$, $(-1, 210^\circ)$, $(-1, -150^\circ)$ all represent the point Q .

To each equation corresponds a single definite curve, but the fact that a given point may be represented by different pairs of coördinates makes it possible that a curve may be represented in the polar system by more than one equation. Thus it is easily seen that the equations $r = 2$, $r = -2$ represent the same curve, a circle of radius 2 with center at O .

Example: Trace the curve

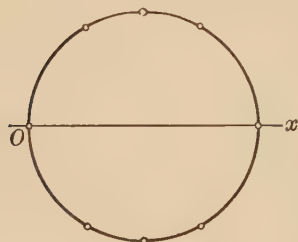


FIG. 84

$$r = 2a \cos \theta.$$

θ	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$
r	$2a$	$\sqrt{3}a$	$\sqrt{2}a$	a	0

Plotting these points, we obtain the upper half of the curve (Fig. 84). Since for values of θ in the *second* quadrant $\cos \theta$, and hence r ,

is negative, the curve falls in the *fourth* quadrant; the values of $\cos \theta$ are numerically the same as in the first quadrant, but in reverse order, so that the lower half of the curve is symmetric to the upper half (see also § 104). In the third quadrant $\cos \theta$ takes the same values as in the first, and in the same order, but negative: the upper half is repeated. Similarly, for θ in the fourth quadrant, the lower half is repeated. Since a complete period of the cosine function has now been covered, values of $\theta > 2\pi$, as well as negative values, will merely repeat the same curve.

103. Analysis of an equation. In the great majority of important polar equations, r is expressed in terms of *trigonometric functions* of θ —commonly sines and cosines. In such cases an analysis similar to that of § 96 would be comparatively ineffective (except as regards tests for symmetry—see § 104). Instead, we proceed as in the above example: that is, we employ our knowledge of trigonometry to *observe the variation of r as θ varies*. Occasionally it may be necessary to plot points through a wide range of values of θ , but as a rule a comparatively narrow interval suffices. Thus in the example of § 102, we plotted points in the first quadrant and from this deduced the entire remainder of the curve.

It thus appears that a thorough knowledge of the fundamentals of trigonometry is indispensable for the work of this chapter, and a review of the subject at this point is strongly advised.

104. Tests for symmetry. While in polar coördinates there is of course no x or y , for brevity we will refer to the polar axis and the perpendicular to this line through O as x -axis and y -axis respectively.

The following theorems are nearly obvious, and may be verified by the reader.

If the equation is unchanged when we change

- | | | |
|--|----------------------------|------------|
| (a) θ to $-\theta$, or | { the curve is symmetric { | x -axis; |
| (b) θ to $\pi - \theta$ and r to $-r$, | | |
| (c) θ to $\pi - \theta$, or | { the curve is symmetric { | y -axis; |
| (d) θ to $-\theta$ and r to $-r$, | | |
| (e) r to $-r$, or | { the curve is symmetric { | origin. |
| (f) θ to $\pi + \theta$, | | |

As a result of the fact that a point may be represented by more than one pair of coördinates (§ 102), the converse theorems are not true. For instance, if the equation is changed when r is replaced by $-r$, the curve still may be symmetric with respect to O , by test (f). As an illustration, see the example of § 105.

Instead of memorizing the above tests, the student should become thoroughly familiar with the geometric meaning of each one, whereupon they will come readily to mind as needed. For instance, test (b) operates as

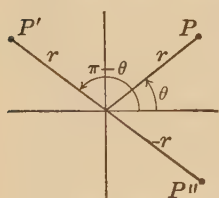


FIG. 85

follows. Changing θ to $\pi - \theta$ reflects a point P to the position P' ; then, changing r to $-r$ reflects P' to P'' , the image of P with respect to Ox .

The tests for symmetry afford a check on the remainder of our analysis, and also act as time-savers, often enabling us to obtain the remainder of the curve by reflection after a limited portion has been plotted. Thus, if a curve is symmetric to both axes, then in most cases we may plot the curve for values of θ between 0° and 90° , and deduce the rest of the curve by reflection in both axes.

However, the qualification “in most cases” must not be overlooked. The above procedure does not always give the entire curve, for various reasons. For instance, values of θ in the third quadrant may give points in the first quadrant not corresponding to first-quadrant values of θ : see, for example, Exs. 11, 12, p. 159. If due care is exercised, such cases are easily recognized.

105. One-valued functions. We consider first the case in which there is one value of r for each value of θ .

Example: Trace the “four-leaved rose” $r = a \cos 2\theta$.

Since $\cos(-2\theta) = \cos 2\theta$, the curve is symmetric to Ox by (a), § 104. Since $\cos 2(\pi - \theta) = \cos(2\pi - 2\theta) = \cos 2\theta$, the curve is symmetric to Oy by (c).

θ	0°	15°	$22\frac{1}{2}^\circ$	30°	45°
2θ	0°	30°	45°	60°	90°
r	a	$\frac{1}{2}\sqrt{3}a$	$\frac{1}{2}\sqrt{2}a$	$\frac{1}{2}a$	0

Plotting the points found above,* we obtain the half-loop numbered 1. As θ ranges from 45° to 90° , 2θ ranges from 90° to 180° : thus the values of $\cos 2\theta$ are numerically the same

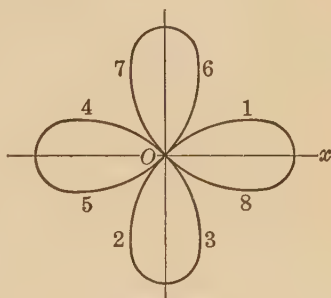


FIG. 86

as those found above, but in reverse order and negative. Since r is negative, the corresponding portion of the curve lies not in the first, but in the third quadrant: the half-loop 2. Reflecting in Oy , we obtain the arcs 3, 4; reflecting in Ox , the balance of the curve. Since this covers a complete “period” of the function $\cos 2\theta$, we have the entire curve.

* It is of course the values of r and θ , not r and 2θ , that are plotted. Thus the second point is $(\frac{1}{2}\sqrt{3}a, 15^\circ)$, etc.

EXERCISES

Plot the following points.

1. $(2, 60^\circ)$, $(3, 180^\circ)$, $(5, 0^\circ)$, $(-5, 0^\circ)$, $(4, -15^\circ)$, $(-1, 15^\circ)$, $(-1, -45^\circ)$, $(-2, 180^\circ)$, $(0, 90^\circ)$, $(1, 220^\circ)$.

2. $(a, \frac{1}{3}\pi)$, $(\frac{1}{2}a, \frac{1}{4}\pi)$, $(2a, \frac{1}{2}\pi)$, $(\frac{1}{2}\sqrt{3}a, \frac{2}{3}\pi)$, $(-\frac{1}{3}a, \frac{3}{4}\pi)$, $(\frac{1}{2}\sqrt{2}a, \frac{5}{4}\pi)$, $(-\frac{1}{2}\sqrt{2}a, -\frac{1}{6}\pi)$, $(a, -\frac{4}{3}\pi)$.

Trace the following curves on polar coördinate paper.

3. $r = a$.

4. $r = -a$.

5. $\theta = -\frac{2}{3}\pi$.

6. $\theta = \frac{1}{6}\pi$.

7. $r = -a \sin \theta$.

8. $r = 2a \sin \theta$.

9. $r = a \cos^2 \theta$.

10. $r = a \sin^2 \theta$.

11. $r = a(1 + \sin^2 \theta)$.

12. $r = a(2 - \cos^2 \theta)$.

13. $r = a(\cos \theta - \sin \theta)$.

14. $r = a \sin 2\theta$.

15. $r(2 - \sin \theta) = a$.

16. $r(2 + \cos \theta) = a$.

17. The limaçon $r = a(\cos \theta + 3)$.

18. The limaçon $r = a(2 - \sin \theta)$.

19. The cardioid $r = a(1 - \sin \theta)$.

20. The cardioid $r = a(\cos \theta + 1)$.

21. The limaçon $r = a(2 \cos \theta - 1)$.

22. The limaçon $r = a(1 + 2 \sin \theta)$.

23. $r = a \sin 3\theta$.

24. $r = a \cos 3\theta$.

25. $r = a \sin^2 2\theta$.

26. $r = a \cos^2 2\theta$.

27. $r = a \cos \theta(1 - \cos \theta)$.

28. $r = a \cos \theta(2 - \sin \theta)$.

29. $r = a \tan \theta$.

30. $r = a \sec^2 \theta$.

In Exs. 31-33, express θ in radians.

31. The spiral of Archimedes $r = a\theta$. Take $a = \frac{1}{\pi}$.

32. The spiral $r = a\theta^2$. Take $a = \frac{1}{\pi^2}$.

33. The hyperbolic spiral $r\theta = a$. Take $a = \pi$.

106. Two-valued functions. Consider the case in which r^2 is expressed as a function of θ .

Since r occurs only in an even power, all curves of this class are symmetric with respect to the pole, by (e), § 104. Hence, if a curve of this class is symmetric to one axis, it must be symmetric to the other also (Ex. 42, p. 22).

Example: Trace the *lemniscate*

$$r^2 = a^2 \cos 2\theta.$$

θ	0	15°	22½°	30°	45°
2θ	0	30°	45°	60°	90°
r^2	a^2	$0.87a^2$	$0.71a^2$	$0.5a^2$	0
r	$\pm a$	$\pm 0.93a$	$\pm 0.84a$	$\pm 0.71a$	0

Since $\cos(-2\theta) = \cos 2\theta$, the curve is symmetric to Ox ; hence also to Oy , by the preceding remark. Plotting the above points, we get the arcs 1, 2. As θ varies from 45° to 90°, 2θ ranges from 90° to 180°, $\cos 2\theta$ is negative, and r is imaginary. Having examined r through the first quadrant, we obtain the balance of the curve by reflection.

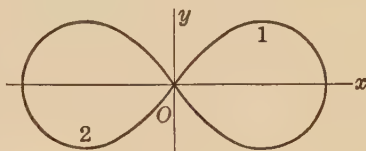


FIG. 87

EXERCISES

Trace the following curves.

1. $r^2 = a^2 \sin \theta$.
2. $r^2 = a^2 \cos \theta$.
3. $r^2 = a^2 \sin 2\theta$.
4. $r^2 = 1 + \sin^2 \theta$.
5. $r^2(1 + \cos^2 \theta) = a^2$.
6. $r^2(1 + 3 \sin^2 \theta) = a^2$.
7. $r^2 \sin 2\theta = a^2$.
8. $r^2 \cos 2\theta = a^2$.
9. $r^2 = 3 - 4 \cos^2 \theta$.
10. $r^2 = 1 - 4 \sin^2 \theta$.
11. $r^2 = a^2(1 + \cos \theta)^3$.
12. $r^2 = a^2(1 - 2 \sin \theta)^3$.
13. $r^2 = a^2(2 - \cos \theta)$.
14. $r^2 = a^2(\sin \theta + 4)$.
15. $r^2 = a^2(1 + \sin \theta)$.
16. $r^2 = a^2(2 \cos \theta - 1)$.
17. $r^2 = a^2 \sin \theta(1 + \sin \theta)$.
18. $r^2 = a^2 \cos \theta(1 - \cos \theta)$.
19. $r^2 = a^2(\sin \theta + \cos \theta)$.
20. $r^2 = a^2 \sin \theta(1 - 2 \sin \theta)$.
21. $r^2 = a^2 \theta$. For convenience, take $a^2 = \frac{1}{\pi}$.
22. $r^2 \theta = a^2$. For convenience, take $a^2 = \pi$.

107. Transformation from one system to the other. The usefulness of polar coördinates lies chiefly in the fact that many important curves are more easily traced, and their properties more easily developed, by using the polar equation of the curve.* Not infrequently both forms are useful for the same curve, some properties appearing more readily from the cartesian, others from the polar form. This suggests the desirability of formulas enabling us to pass from either system to the other.

Let the point P (Fig. 88) have the cartesian coördinates x, y and the polar coördinates r, θ . Then it is obvious that

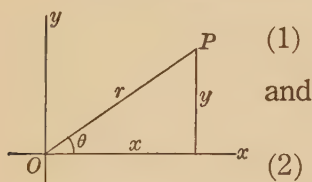


FIG. 88

$$\begin{cases} x = r \cos \theta, \\ y = r \sin \theta, \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2}, \\ \tan \theta = \frac{y}{x}. \end{cases}$$

Examples: (a) Obtain the equation of the circle $x^2 + y^2 = 2ax$ in polar coördinates.

Since $x^2 + y^2 = r^2$, we have at once

$$r^2 = 2ar \cos \theta,$$

$$r = 2a \cos \theta.$$

(b) Change the equation $r = 2a \sin \theta$ to cartesian coördinates.

While not necessary, it is most convenient first to multiply both members by r , to introduce the combination $r \sin \theta$:

$$r^2 = 2ar \sin \theta.$$

Then we have at once

$$x^2 + y^2 = 2ay.$$

* As an illustration, the student might try to solve the example of § 105, using the cartesian equation $(x^2 + y^2)^3 = a^2(x^2 - y^2)^2$.

EXERCISES

Assume that the axes are placed as in Fig. 88.

1. Find the cartesian coördinates of the points (a) $(1, 30^\circ)$; (b) $(-1, \frac{1}{4}\pi)$; (c) $(-2, 0)$; (d) $(3, -60^\circ)$; (e) $(-2, \frac{5}{8}\pi)$; (f) $(3, 225^\circ)$.

2. Find the polar coördinates of the points (a) $(2, 2)$; (b) $(\sqrt{3}, -1)$; (c) $(0, 0)$; (d) $(-3, -4)$; (e) $(-5, -5)$; (f) $(0, -3)$; (g) $(-1, 0)$.

Find the equations of the following curves in polar coördinates.

3. $y = x$.

4. $x^2 + y^2 = a^2$.

5. $y = 2x + 3$.

6. $x + y = 1$.

7. $y = a$.

8. $x = a$.

9. $x^2 = 4ay$.

10. $y^2 = 4a(x - a)$.

11. $(x^2 + y^2)^2 = ay^3$.

12. $(x^2 + y^2)^2 = 8x$.

13. $y = x^3$.

14. $y^2 = x^3$.

15. $x \cos \beta + y \sin \beta = p$. (§ 33.)

Ans. $r \cos(\theta - \beta) = p$.

16. $x^4 - y^4 = 2a^2xy$.

Ans. $r^2 = a^2 \tan 2\theta$.

Find the equations of the following curves in cartesian coördinates.

17. $r = a$.

18. $\theta = 45^\circ$.

19. $\theta = 0$.

20. $\tan \theta = 2$.

21. $\sin \theta = \frac{3}{5}$.

22. $\cos \theta + \frac{4}{5} = 0$.

23. $r = 2 \csc \theta$.

24. $r + 3 \cos \theta = 0$.

25. $r = \sec \theta \tan \theta$.

26. $r^2 \cos 2\theta = a^2$.

27. $r \cos(\theta - \frac{1}{4}\pi) = \sqrt{2}$.

Ans. $x + y = 2$.

28. $\sqrt{2}r \sin(\theta - \frac{1}{4}\pi) = 1$.

Ans. $x - y + 1 = 0$.

29. $r = a(1 - \cos \theta)$.

Ans. $(x^2 + y^2 + ax)^2 = a^2(x^2 + y^2)$.

108. Locus problems in polar coördinates. To find the equation of a locus, using polar coördinates, the procedure is exactly the same as in §§ 21, 22. That is, we assume a point $P : (r, \theta)$ in a general position on the curve, and from the statement of the problem or by means of some characteristic property of the figure try to obtain an equation involving r, θ , and the constants of the problem: this equation must be the equation of the given locus.

Polar coördinates are strongly indicated in that numerous class of problems where the distance of the moving

point from a fixed point varies according to some simple law. The fixed point should be taken as pole and a line of symmetry (if such exists) as polar axis.

Example: In Fig. 89, a random line is drawn through O intersecting the circle at P ; P is projected to M ; a length $OQ = OM$ is laid off on OP . Find the locus of Q .

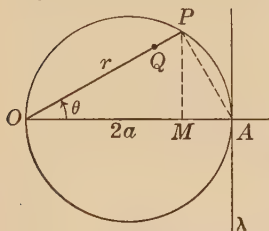


FIG. 89

By trigonometry,

$$OP = 2a \cos \theta,$$

$$OM = OP \cos \theta = 2a \cos^2 \theta;$$

since

$$OM = OQ = r,$$

$$r = 2a \cos^2 \theta. \quad (\text{Ex. 9, p. 158.})$$

EXERCISES

Find the polar equations of the following loci. Draw the figures.

1. A circle of radius 5 with center at O .
2. A line through O making an angle of 45° with Ox .
3. A circle of radius a with center at $(a, 0)$. (Fig. 84.)
4. A circle of radius a with center at $(a, \frac{1}{2}\pi)$.
5. A line through $(a, 0)$ perpendicular to Ox .
6. A line through $(a, \frac{1}{2}\pi)$ parallel to Ox .
7. The path of a point which is equidistant from a given point and a given line. Take the given point as pole and the given line perpendicular to Ox at $(a, 0)$.
Ans. $r(1 + \cos \theta) = a$.
8. Ex. 7, if the moving point is half as far from the given point as from the given line.
Ans. $r(2 + \cos \theta) = a$.
9. Ex. 7, if the moving point is twice as far from the given point as from the given line.
Ans. $r(1 + 2 \cos \theta) = 2a$.
10. Ex. 7, taking the given line parallel to Ox through $(a, \frac{1}{2}\pi)$.
Ans. $r(1 + \sin \theta) = a$.
11. In Fig. 89, a distance $OR = MA$ is laid off on OP . Find the locus of R . (Ex. 10, p. 158.)
12. In Fig. 89, a distance $OS = OM - MA$ is laid off on OP . Show that the locus of S is a "four-leaved rose" (§ 105).

13. In Fig. 89, the line OP is produced to a point R such that $PR = 2a$. Show that the locus of R is a cardioid. (Ex. 20, p. 158.)

14. In Fig. 89, the line OP is produced to R such that $PR = AP$. By transforming the answer to cartesian coördinates, show that the locus of R is a circle.

15. In Fig. 89, a point R is marked on OP such that $OR = OP - OM$. Find the locus of R . (Ex. 27, p. 158.)

16. In Fig. 89, a distance $AT = AP$ is laid off on the vertical line λ ; a distance $OR = OT$ is then laid off on OP produced. Find the locus of R . (Ex. 4, p. 159.)

17. In Fig. 89, a point R is marked on OP such that $OR = MP$. Find the locus of R . (Ex. 14, p. 158.)

18. In Fig. 89, let OP produced intersect λ at a point T : find the locus of the midpoint of PT , and trace the curve. *Ans.* $r = a(\sec \theta + \cos \theta)$.

19. In Fig. 89, find the locus of the midpoint of AP ; trace the curve.

$$\text{Ans. } r^2 = 3ar \cos \theta - 2a^2.$$

20. A point moves so that the product of its distances from two fixed points is one-fourth the square of the distance between the points. Using cartesian coördinates, and taking the points $(\pm b, 0)$, find the equation of the locus. Show by transformation that the locus is a lemniscate (see the example, § 106).

109. Polar equation of a straight line. If a straight line passes through O , its polar equation may be written at once as

$$\theta = \alpha,$$

where α is the inclination of the line (§ 8).

If the line does not pass through the origin, let N be the foot of the perpendicular from the origin to the line, and let the polar coördinates of N be (p, β) . Then in the triangle OPN we have at once

$$\cos(\theta - \beta) = \frac{p}{r},$$

or

$$r \cos(\theta - \beta) = p.$$

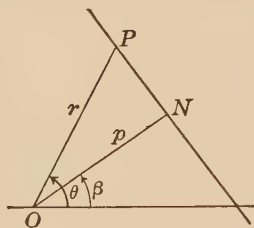


FIG. 90

Evidently this is the polar equivalent of the normal form (§ 33), as may be verified by transforming either equation into the other. (Cf. Ex. 15, p. 161.)

110. Polar equation of a conic. To find the equation of a conic in polar coördinates, let us take the focus as pole and the line through the focus perpendicular to the directrix λ as polar axis. Denote by $2l$ the length of the latus rectum Q_1Q_2 ; then the distance OB from the focus to the directrix is $\frac{l}{e}$. For, the point Q_1 is a point on the curve; hence, by the definition of § 44,

$$OQ_1 = e \cdot Q_1M,$$

so that

$$OB = Q_1M = \frac{OQ_1}{e} = \frac{l}{e}.$$

Now assume a point $P : (r, \theta)$ in a general position on the curve, and drop a perpendicular PL to the directrix. Since

$$OP = e \cdot PL,$$

it follows that

$$PL = \frac{r}{e}.$$

Also

$$ON = r \cos \theta.$$

Evidently

$$ON + PL = OB,$$

or

$$r \cos \theta + \frac{r}{e} = \frac{l}{e}.$$

This equation when solved for r takes the form

$$(1) \quad r = \frac{l}{1 + e \cos \theta}.$$

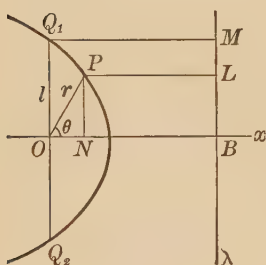


FIG. 91

EXERCISES

Derive the equations of the following conics in polar coördinates, taking the focus as pole in each case.

1. With $e = 1$ and directrix perpendicular to Ox through $(3, \pi)$.

$$\text{Ans. } r = \frac{3}{1 - \cos \theta}.$$

2. With $e = \frac{1}{2}$ and directrix perpendicular to Ox through $(2, 0)$.

$$\text{Ans. } r = \frac{2}{2 + \cos \theta}.$$

3. With $e = 4$ and directrix parallel to Ox through the point $(3, \frac{1}{2}\pi)$.

$$\text{Ans. } r = \frac{12}{1 + 4 \sin \theta}.$$

4. With $e = \frac{4}{5}$ and directrix parallel to Ox through $(2, \frac{1}{2}\pi)$.

$$\text{Ans. } r = \frac{8}{5 + 4 \sin \theta}.$$

5. By transformation of equation (1), § 52, find the polar equation of a central conic with center at the pole and transverse axis in Ox .

6. Derive the polar equation of a parabola with vertex at the pole and focus at $(a, 0)$, (a) directly, (b) by transforming the equation $y^2 = 4ax$.

$$\text{Ans. } r \sin \theta \tan \theta = 4a.$$

7. Show that the locus of the midpoints of the focal radii of any conic is a conic of the same shape (§ 67) but half the size of the original conic.

8. Chords are drawn through the vertex of a parabola. Show that the locus of their midpoints is a parabola half the size of the original—i.e. with latus rectum half as great. (Use the answer to Ex. 6.)

9. Find the locus of the midpoints of chords drawn through a fixed point on a circle.

10. Given the axis, focus, and one point of a parabola, construct the directrix by ruler and compass, using the property (1), § 110.

11. Solve Ex. 10 another way.

12. Given the axis, vertex, and one other point of a parabola, construct the focus. (Use the answer to Ex. 6.)

13. Using (1), § 110, prove the theorem of § 67.

14. Using (1), § 110, prove the converse of the theorem of § 67.

15. Prove that when two parabolas have a common focus and open in the same direction, they are in perspective (§ 65), with the focus as center of similitude. (Ex. 13.)

16. Solve Ex. 15, reading "vertex" instead of "focus." (Ex. 6.)

CHAPTER XII

PARAMETRIC EQUATIONS

111. Parametric equations. For many loci, it is difficult to obtain either the cartesian or the polar equation directly; instead, the definition leads naturally to *two* equations giving x and y respectively in terms of some third variable. This auxiliary variable is called a *parameter*, and the two equations are *parametric equations* of the curve. To obtain the ordinary cartesian equation we have merely to *eliminate the parameter between the two equations*.

The cartesian equation of a curve is unique: for a given curve in a given position on the axes there can be only one equation.* As regards parametric representation, the case is quite otherwise: two different problems may lead, each in a perfectly natural way, to two different pairs of parametric equations for the same curve. For instance, if a point moves in a plane in a certain way, the coördinates of the moving point at any time t may be given by the equations

$$(1) \quad x = t, \quad y = 1 - t.$$

Under another law of motion the equations may be

$$(2) \quad x = \cos^2 t, \quad y = \sin^2 t.$$

Adding either equations (1) or equations (2), we get

$$(3) \quad x + y = 1,$$

which shows that in each case the point moves in the straight line (3).

* While it is possible to find fault with this remark, the objections are purely artificial: for all practical purposes, the statement is strictly true. See the footnote, p. 40.

Sometimes the parametric equations are merely incidental, being discarded entirely as soon as the cartesian equation is obtained. In other cases the cartesian equation is so complicated that the problem is best studied by means of the parametric equations (see, for example, § 121). Again, it may happen that both forms are essential. Thus, in the first motion-problem above, it appears from (3) that the point moves in a straight line. Since, by (1), both x and y are linear functions of t , it follows at once (see § 31) that the point moves with constant speed—a fact which is not shown at all by (3).

112. Representation of a portion of a curve. In the elementary applications the parameter is of course restricted to real values, and frequently, by the very nature of the problem, is still further restricted, for example to positive values. (Compare the last paragraph of § 30.) Because of these restrictions, the parametric equations in many cases represent only a portion of the curve whose cartesian equation is found by elimination.* Thus, if t may assume any real value, equations (1) of § 111 represent the entire line; equations (2), only the segment in the first quadrant, since x and y must be positive. In studying the motion of a point, we would as a rule consider the motion as starting at time $t = 0$, and would be interested only in subsequent—i.e. positive—values of t ; if so, equations (1) represent only the part of the line to the right of the y -axis, since x must be positive.

113. Parametric equations of the ellipse. The circle described on the major axis of an ellipse as a diameter is called the *major auxiliary circle*; that on the minor axis is

* Such a phenomenon is not surprising. If we prove that every (x, y) -pair satisfying two parametric equations also satisfies a certain cartesian equation, it does not follow, conversely, that every pair satisfying the cartesian equation must satisfy the parametric equations.

the *minor auxiliary circle*. Evidently, for the ellipse

$$(1) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the equations of these circles are respectively

$$x^2 + y^2 = a^2, \quad x^2 + y^2 = b^2.$$

If the ordinate MP of a point P on an ellipse be produced until it cuts the major auxiliary circle at a point Q , and a line OQ be drawn from the center to Q , the angle $\phi = \angle MOQ$ is called the *eccentric angle* of the point P .

Evidently, in Fig. 92,

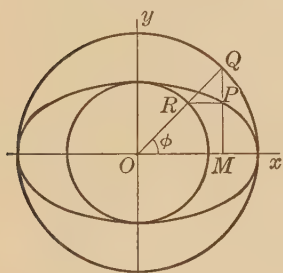


FIG. 92

$$OM = OQ \cos \phi;$$

that is, since $OM = x$, $OQ = a$,

$$x = a \cos \phi.$$

Substituting this value of x in (1) and solving for y^2 , we get

$$y^2 = b^2(1 - \cos^2 \phi),$$

or

$$y = b \sin \phi.$$

It follows from this that the line OQ and the line through P parallel to the major axis intersect the minor auxiliary circle in the same point R .

The equations

$$(2) \quad x = a \cos \phi, \quad y = b \sin \phi$$

are *parametric equations of the ellipse* in terms of the eccentric angle. For, if we write (2) in the form

$$\frac{x}{a} = \cos \phi, \quad \frac{y}{b} = \sin \phi,$$

square both members of each equation, and add, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \phi + \sin^2 \phi = 1.$$

114. Point-plotting from parametric equations. To plot, by points, a curve represented by parametric equations, we merely assign suitable values to the parameter and compute the corresponding values of x and y .

Example: Plot the curve

$$(1) \quad x = a \cos^3 \theta, \quad y = a \sin^3 \theta.$$

θ	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$
x	a	$\frac{3}{8}\sqrt{3}a$	$\frac{1}{4}\sqrt{2}a$	$\frac{1}{8}a$	0
y	0	$\frac{1}{8}a$	$\sqrt{2}a$	$\frac{3}{8}\sqrt{3}a$	a

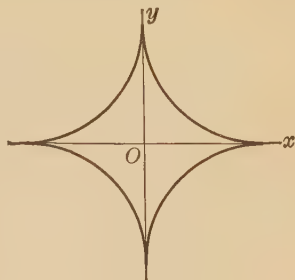


FIG. 93

Plotting these points, we get the portion of Fig. 93 lying in the first quadrant. By taking the cube root of the square of each member, the given equations may be written in the form

$$(2) \quad x^{2/3} = a^{2/3} \cos^2 \theta, \quad y^{2/3} = a^{2/3} \sin^2 \theta;$$

adding, we get the cartesian equation

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

From this it appears that the curve is symmetric to both axes, whence the balance of the curve may be obtained by reflection.

The student is warned not to confuse parametric representation with polar coördinates. The parameter θ here occurring is quite different from the polar angle θ . To make sure of this, divide the second of equations (1) by the first, member by member:

$$\tan^3 \theta = \frac{y}{x}, \quad \tan \theta = \frac{y^{1/3}}{x^{1/3}}.$$

For the polar θ , we have

$$\tan \theta = \frac{y}{x}.$$

EXERCISES

Find the cartesian equation of the given curve, and draw the curve from the cartesian equation. In case the cartesian and parametric representations are not fully equivalent, determine what part of the curve is represented by the given equations.

1. $x = \sec^2 \phi$, $y = -\tan^2 \phi$.
2. $x = 2 - 3t$, $y = 3t - 1$.
3. $x = 1 - 2t$, $y = 3t - 2$.
4. $x = 4t + 6$, $y = 2t - 3$.
5. $x = a \sin 2\phi$, $y = a \cos 2\phi$.
6. $x = 4 \sin \phi$, $y = 3 \cos \phi$.
7. $x = \sin \theta$, $y = \cos 2\theta$.
8. $x = 2 \cos 2\theta$, $y = \cos \theta$.
9. $x = t^2$, $y = 1 - t$.
10. $x = t + 2$, $y = t^2 - 1$.
11. $x = a \tan^2 \theta$, $y = 2a \tan \theta$.
12. $x = a \sin \phi$, $y = a \csc \phi$.
13. $x = a \sec \phi$, $y = b \tan \phi$.
14. $x = a \cot \phi$, $y = a \csc \phi$.
15. $x = \frac{1}{2} \sin 2\alpha$, $y = \sin^2 \alpha$.
16. $x = \cos^2 \alpha$, $y = \sin 2\alpha$.
17. $x = 1 + t^2$, $y = \frac{1 + t^2}{2t + t^2}$.
18. $x = 1 - t$, $y = \frac{t^2 - 1}{t^2}$.

In the following cases, obtain the cartesian equation; plot the curve (or part-curve) by points, using the parametric equations.

19. $x = 1 + t^2$, $y = 4t - t^3$. *Ans.* $y^2 = (x - 1)(x - 5)^2$.
20. $x = 1 - t^2$, $y = t + t^3$. *Ans.* $y^2 = (1 - x)(2 - x)^2$.
21. $x = \sin \phi + \cos \phi$, $y = \sin \phi$. *Ans.* $x^2 - 2xy + 2y^2 = 1$.
22. $x = \sin \phi + \cos \phi$, $y = \sin \phi - \cos \phi$. *Ans.* $x^2 + y^2 = 2$.
23. $x = \frac{1}{1 + t}$, $y = \frac{2}{1 + t^2}$. *Ans.* $y = \frac{2x^2}{2x^2 - 2x + 1}$.
24. $x = \frac{1}{1 + t}$, $y = \frac{1}{1 - t^2}$. *Ans.* $x^2 - 2xy + y = 0$.
25. Draw a curve from which $\sin 2\theta$ may be read off if $\sin \theta$ is given.
($x = \sin \theta$, $y = \sin 2\theta$.) *Ans.* $y^2 = 4x^2(1 - x^2)$.
26. Draw a curve from which $\sin \theta$ may be read off if $\tan \theta$ is given.
Ans. $y^2 = \frac{x^2}{1 + x^2}$.

CHAPTER XIII

SPECIAL CURVES

115. The strophoid. In this chapter we discuss briefly a number of well-known curves of historical interest or practical importance.

In Fig. 94, the line AB rotates about A , intersecting the fixed line λ at B . On AB points P , P' are marked such that $BP = BP' = OB$. The locus of P and P' is a *strophoid*.

To find the equation of the strophoid, the second paragraph of § 108 suggests polar coördinates with A as pole and AO as axis. With this set-up, it is obvious almost at once that the locus of P is

$$(1) \quad r = a(\sec \theta - \tan \theta).$$

It might seem that, for the locus of P' , the equation

$$(2) \quad r = a(\sec \theta + \tan \theta)$$

would have to be used; but it is easily verified by plotting that, as θ varies from 0 to 2π , either equation gives the entire curve, so that only one need be retained.*

* An analytic proof of this statement: If, in either equation, we replace θ by $\pi + \theta$ and r by $-r$, the other equation is obtained; but the coördinates (r, θ) and $(-r, \pi + \theta)$ represent the same point. This is merely another instance of the fact that two different polar equations may represent the same curve.

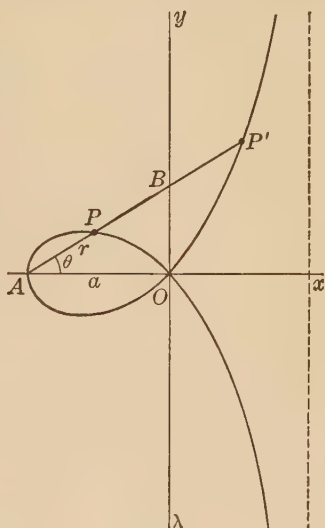


FIG. 94

In cartesian coördinates *with* O as origin either of the above equations becomes

$$(3) \quad y^2 = \frac{x^2(a+x)}{a-x}.$$

A point where there are two distinct tangents to a curve is called a *node*. Evidently the strophoid has a node at O . (See also Fig. 87, p. 159.)

116. The cissoid. In Fig. 95, the line OQ rotates about O , intersecting the fixed circle at M and the fixed line λ at Q . On OQ a point P is marked such that $OP = MQ$. The locus of P is a *cissoid*.

The polar equation of the cissoid is

$$(1) \quad r = a(\sec \theta - \cos \theta),$$

which is equivalent to

$$(2) \quad r = a \sin \theta \tan \theta;$$

in cartesian coördinates,

$$(3) \quad y^2 = \frac{x^3}{a-x}.$$

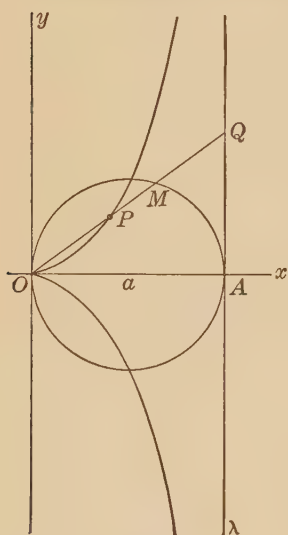


FIG. 95

In (2), $r = 0$ when $\theta = 0$, and when θ is a small angle either positive or negative, r is small and positive. It thus appears that the

curve comes in to the origin tangent to Ox , and turns back tangent to the same line. A point where the curve behaves in this way is called a *cusp*.

117. The trisectrix of Maclaurin. In Fig. 96, let $OA = a$, $OB = 4a$. The line OC , rotating about O , intersects the circle in C and the line λ in M . On OC a point

P is marked such that $OP = MC$. The locus of P is the *trisectrix of Maclaurin*.

Since

$$OC = 4a \cos \theta,$$

$$OM = a \sec \theta,$$

we have at once

$$(1) \quad r = a(4 \cos \theta - \sec \theta),$$

or in cartesian coördinates,

$$(2) \quad y^2 = \frac{x^2(3a - x)}{a + x}.$$

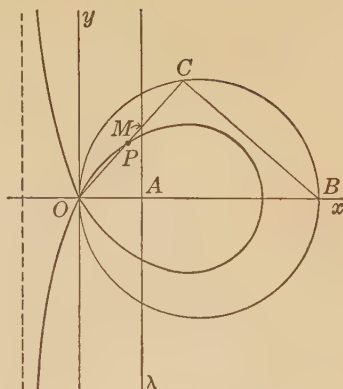


FIG. 96

118. The conchoid. In Fig. 97, the line OB intersects the fixed line λ at B ; in each direction from B a distance b is laid off. As OB rotates, the locus of P and P' is a *conchoid*.

Since $OB = a \sec \theta$, we have at once

$$(1) \quad r = a \sec \theta - b.$$

(The equation

$$r = a \sec \theta + b$$

is equivalent to (1): cf. the discussion of a similar situation in § 115.) In cartesian coördinates, this becomes

$$(2) \quad (x - a)^2(x^2 + y^2) = b^2x^2.$$

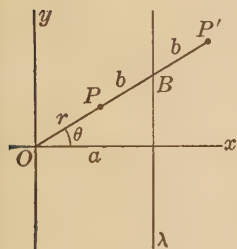


FIG. 97

Setting $r = 0$ in (1), we find $\sec \theta = \frac{b}{a}$, or $\cos \theta = \frac{a}{b}$.

For $b > a$, this gives two distinct values of $\theta < \pi$, and the origin is a node. When $b = a$ we get $\theta = 0, \theta = \pi$; these directions coincide and the origin is a cusp. When $b < a$, θ is imaginary and the origin is not on the curve.*

* Even in this case (2) is satisfied by $(0, 0)$. This point is extraneous, being introduced in the process of transforming from polar to cartesian. Cf. § 19.

119. The limaçons and the cardioid. The straight line OB , rotating about O , meets the circle at B ; points P and P' are marked such that $BP = BP' = b$, a constant. The locus of P and P' is a *limaçon*.

The polar equation is obviously

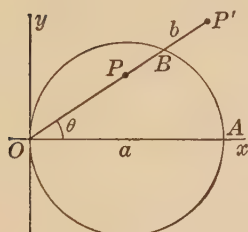


FIG. 98

$$(1) \quad r = a \cos \theta - b$$

(or the equivalent

$$r = a \cos \theta + b;$$

see § 115). In cartesian coördinates (1) becomes

$$(2) \quad (x^2 + y^2 - ax)^2 = b^2(x^2 + y^2).$$

The origin is a node if $b < a$; a cusp if $b = a$; not on the curve if $b > a$. When $b = a$, the curve is called a *cardioid*. Curves of each type were traced in Exs. 17–22, p. 158.

120. The unsolved problems of antiquity. The ancient Greek geometers spent much time, without success, in trying to solve by ruler and compass three celebrated problems:

- (1) To “duplicate the cube”—i.e. to construct a cube whose volume shall be twice the volume of a given cube.
- (2) To trisect an arbitrary angle.
- (3) To “square the circle”—i.e. to construct a square of area equal to that of a given circle.

By way of making at least some progress, the Greeks and later mathematicians evolved a number of higher plane curves with whose help the first two problems can be solved. Thus the cissoid was invented by Diocles for the purpose of duplicating the cube. The construction is as follows.

Let $OA = a$, the edge of the cube to be “duplicated.”

Lay off $OB = 2a$; draw AB intersecting the cissoid at P . Draw and produce OP , meeting the asymptote at Q ; then $AQ = 2^{1/3}a$, so that a cube of edge AQ will have volume $2a^3$. An analytic proof:*

The equation of AB is evidently

$$\frac{x}{a} + \frac{y}{2a} = 1.$$

Solving this equation with that of the cissoid, we find the coördinates of P to be $\left(\frac{2^{2/3}}{1 + 2^{2/3}} a, \frac{2}{1 + 2^{2/3}} a \right)$, so that the equation of OP is

$$y = 2^{1/3}x.$$

When $x = a$, $y = AQ = 2^{1/3}a$.

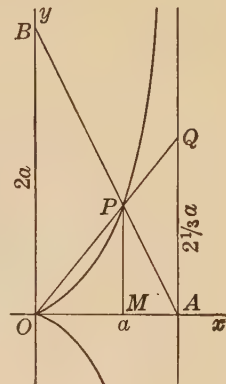


FIG. 99

The conchoid was invented by Nicomedes for the purpose of trisecting any angle; it can also be used to duplicate the cube. An angle can be trisected by the trisectrix of Maclaurin (whence the name), also by the limaçon for which $a = 2b$. It is not difficult to devise other curves for solving these problems.

It is now well known that none of these problems can be solved by the classic method of ruler-and-compass. For the third problem, even such solutions as those just discussed are impossible: no algebraic curve whose equation has rational coefficients will suffice to square the circle.

* The discoverer of the curve of course gave a purely geometric proof. For this, nothing more is required than the properties of similar triangles, together with the characteristic property of the cissoid expressed directly by its equation, viz. that

$$\frac{\overline{MP}^2}{\overline{OM}^2} = \frac{OM}{MA}.$$

EXERCISES

1. Derive equation (1), § 115.
2. Transform equation (1), § 115, to cartesian coördinates; by translating the origin from A to O (§ 62), derive equation (3). Draw the curve from this equation (§ 99).
3. Transform equation (3), § 115, to polar coördinates. From this result, show that the tangents to the strophoid at the node make angles of $\pm 45^\circ$ with Ox . *Ans.* $r \cos \theta = -a \cos 2\theta$.
4. In Fig. 89, p. 162, find the locus of the point of intersection of AP with the line bisecting $\angle AOP$. By transforming the answer to cartesian coördinates and comparing with (3), § 115, or by direct comparison with Ex. 3, show that the locus is a strophoid. *Ans.* $r \cos \theta = 2a \cos 2\theta$.
5. By using (1), § 115, show that, in Fig. 94, $PP' = 2a \tan \theta$. Hence show that, if AP' is produced to meet the asymptote at a point N , the ordinate of N is equal to PP' .
6. In Fig. 94, prove that the triangles AOP , AOP' are similar.
 (Show that $AP \cdot AP' = \overline{AO}^2$, so that $\frac{AP}{AO} = \frac{AO}{AP'}$)
7. In Fig. 94, show that $\angle POP'$ is a right angle. (Note the position of B relative to O , P , P' .)
8. Prove the theorem of Ex. 7 analytically. (Find OP and OP' by the Cosine Law; then show that $\overline{PP'}^2 = \overline{OP}^2 + \overline{OP'}^2$.)
9. Derive equations (1) and (2), § 116.
10. Derive (3), § 116, by transformation. Draw the curve (§ 99).
11. Find the points of intersection of the cissoid and the circle, (a) by solution of their equations; (b) directly from the construction.
12. In Fig. 94, draw a line through O parallel to AP' , and drop a perpendicular from B to this line, intersecting it at Q . Taking the pole at O , find the locus of Q . (Drop a perpendicular from O to AP' intersecting it at R ; note that $OQ = AB - AR$.) *Ans.* The cissoid $r = a(\sec \theta - \cos \theta)$.
13. In Fig. 47, p. 78, draw OP ; on OP construct a point Q such that $OP \cdot OQ = a^2$. Show that the locus of Q is a cissoid. (Ex. 6, p. 165, and equation (2), § 116.)
14. Derive (2), § 117, by transformation of (1), and trace the curve.
15. For the trisectrix, find the inclinations of the nodal tangents.
16. Derive (2), § 118. Trace the curve for $b = 2a$, finding the slopes of the nodal tangents from the polar equation.
17. Derive (2), § 119.

18. The cissoid of § 116 and the limaçon of § 119 are drawn on the same axes; on each line radiating from O a point P is determined by adding the radius vectors of the two curves. Show that the locus of P is a conchoid.

19. If the two curves of Ex. 18 are the cissoid of § 116 and the trisectrix of § 117, show that the locus of P is a circle.

20. If the two curves of Ex. 18 are the conchoid of § 118 and the trisectrix of § 117, show that the locus of P is a limaçon. For what value of b will the locus be a cardioid? *Ans.* $b = 4a$.

21. Show that chords drawn through the cusp of a cardioid are of constant length $2a$.

22. Carry out the details of proof for the construction in § 120.

23. Show that the cissoid can be used to construct the cube root of any rational number k . (Taking $a = 1$, lay off $OB = k$ and carry out the same construction and proof as in § 120, showing that $AQ = k^{1/3}$.)

24. Following the method of Ex. 23, show that the curve $y^{n-1} = \frac{x^n}{1-x}$ can be used to construct the n -th root of k .

25. Show how to trisect any angle, using the curve $y = 3x - 4x^3$. (See Ex. 31, p. 146.) Plot the curve accurately in the interval $-1 < x < 1$ [note that since y is a sine, the highest point is $(\frac{1}{2}, 1)$], and use it to trisect (a) 60° , (b) 150° , (c) an arbitrary angle in the third quadrant.

121. The cycloid. When a circle rolls, without slipping, on a straight line, the path of any point P of the circle is called a *cycloid*. (See Fig. 100.)

Parametric equations of the cycloid are easily found in terms of θ , the angle through which the circle has turned at any instant. First we recall that, by definition, the radian measure of an angle is the ratio of the arc to the radius:

$$\text{angle} = \frac{\text{arc}}{\text{radius}}, \quad \text{or} \quad \text{arc} = \text{radius} \times \text{angle}.$$

Hence, in Fig. 100,

$$\text{arc } PN = a\theta.$$

But if P was originally at O , then arc PN is equal to the distance the circle has rolled:

$$\text{arc } PN = ON = a\theta.$$

It now appears very readily from the figure that

$$(1) \quad \begin{cases} x = a(\theta - \sin \theta), \\ y = a(1 - \cos \theta). \end{cases}$$

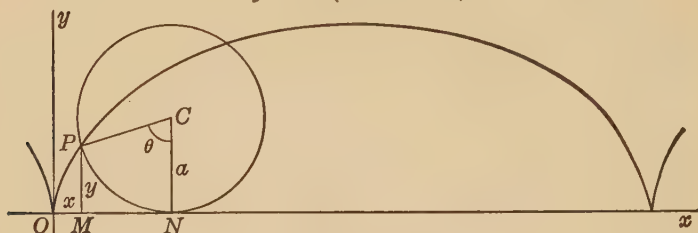


FIG. 100

122. The epicycloid. When a circle rolls, without slipping, on the *outside* of a fixed circle, the path of any point of the rolling circle is called an *epicycloid*.

If the moving point P started at A , then in Fig. 101,

arc $AT = \text{arc } TP$,

or

$$(1) \quad a\theta = b\phi.$$

Since

$$\begin{aligned} x &= OM = ON + NM, \\ y &= MP = NC - QC, \end{aligned}$$

we read off at once

$$(2) \quad \begin{cases} x = (a + b) \cos \theta + b \cos \alpha, \\ y = (a + b) \sin \theta - b \sin \alpha. \end{cases}$$

But $\alpha = \pi - (\theta + \phi)$; substituting in (2), we find

$$\begin{aligned} x &= (a + b) \cos \theta - b \cos (\theta + \phi), \\ y &= (a + b) \sin \theta - b \sin (\theta + \phi). \end{aligned}$$

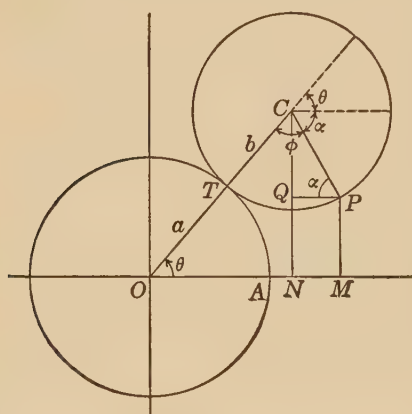


FIG. 101

Substituting the value of ϕ from (1), we obtain finally

$$(3) \quad \begin{cases} x = (a + b) \cos \theta - b \cos \frac{a + b}{b} \theta, \\ y = (a + b) \sin \theta - b \sin \frac{a + b}{b} \theta. \end{cases}$$

Denote the ratio of a to b by n :

$$a = nb.$$

If n is an integer, then after one trip around the fixed circle the point P returns to the starting point A , having traced out a curve of n arches and n cusps (since the circumference of the fixed circle is n times that of the rolling circle). Figure 102 shows one-third of the three-cusped epicycloid $a = 3b$. If n is a rational fraction, proper or improper, P will ultimately return to A , but if n is irrational (i.e. if a and b are incommensurable) P will never return exactly to the starting point.

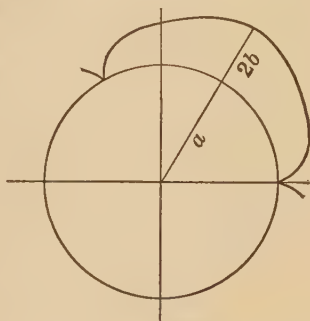


FIG. 102

123. The hypocycloid. When a circle rolls, without slipping, on the *inside* of a fixed circle, the path of any point of the rolling circle is a *hypocycloid*. The equations are obtained from (3), § 122, by replacing b by $-b$:

$$(1) \quad \begin{cases} x = (a - b) \cos \theta + b \cos \frac{a - b}{b} \theta, \\ y = (a - b) \sin \theta - b \sin \frac{a - b}{b} \theta. \end{cases}$$

The hypocycloid of four cusps, $a = 4b$, is shown in Fig. 93, p. 169.

124. The involute of a circle. Let A and Q be respectively a fixed point and a moving point on a fixed circle.

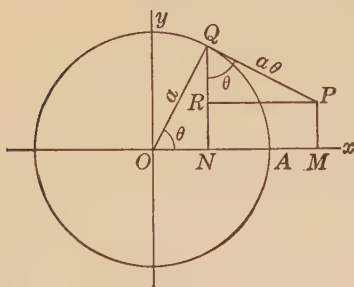


FIG. 103

If a distance $PQ = \text{arc } AQ$ is laid off on the tangent at Q , the locus of P is the *involute of the circle*.

The equations of the curve in terms of the central angle θ may be read off almost at once:

$$(1) \quad \begin{cases} x = a \cos \theta + a\theta \sin \theta, \\ y = a \sin \theta - a\theta \cos \theta. \end{cases}$$

EXERCISES

1. Show in detail the derivation of equations (1), § 121.
2. Show in detail the derivation of equations (2), § 122.
3. Prove that the epicycloid of one cusp ($a = b$) is a cardioid (§ 119).
4. Plot the two-cusped epicycloid ($a = 2b$).
5. Plot the epicycloid $b = 2a$.
6. Prove that when $a = 2b$, the hypocycloid is merely a diameter of the fixed circle.
7. Prove that for the hypocycloid of four cusps ($a = 4b$), equations (1) of § 123 become $x = a \cos^3 \theta$, $y = a \sin^3 \theta$. (Cf. the example of § 114.)
8. Show in detail the derivation of (1), § 124.
9. Plot the involute of the circle in the interval from $\theta = 0$ to $\theta = \pi$.
10. On OQ or OQ produced (Fig. 103) a point S is marked such that $OS = QP$. Show that the locus of S is a spiral of Archimedes (Ex. 31, p. 158).
11. On OQ and OQ produced (Fig. 103), points S, S' are marked such that $QS = QS' = QN$. Show that the locus of S and S' is a cardioid.
12. On OQ and OQ produced (Fig. 103), points S, S' are marked such that $QS = QS' = ON + NQ$. Noting that

$$\frac{1}{2}\sqrt{2}(\cos \theta + \sin \theta) = \cos(\theta - 45^\circ),$$
 show that the locus of S and S' is a limaçon symmetric to the line $\theta = 45^\circ$.

CHAPTER XIV

CURVE-FITTING

125. Empirical equations. In many of the applications of mathematics we are concerned with a set of pairs of values of two variables as determined by observation or experiment. Examples may be cited from a great variety of fields:

(1) Physics: The attraction between two electrified bodies, as a function of the distance between them.

(2) Chemistry: The amount of a given element present in a compound during a chemical reaction, as a function of the time.

(3) Mechanical Engineering: The gasoline consumption of an automobile, as a function of the speed.

(4) Biology: The rate of growth of a culture of bacteria, as a function of time.

(5) Economics: The earning power of a man, as a function of his age.

(6) Sociology: The number of deaths from tuberculosis in a community per annum, as a function of the amount of money spent on preventive activities.

(7) Agriculture: The yield of a piece of land, as a function of the amount of fertilizer used.

By plotting the observed pairs of values on cartesian coördinate paper and drawing a smooth curve through the points, we obtain a more or less accurate graphic representation of the function in hand. The problem of this chapter is to obtain an analytic representation—i.e. to obtain an *equation* which shall correspond to the

empirically-determined curve. An equation representing a set of experimental data is called an *empirical equation*, or *empirical formula*, and the process of finding such an equation is called *curve-fitting*.

According to the definition of "locus of an equation" (§ 13)—a definition to which we have strictly adhered up to this point—the correspondence between a curve and its equation is *unique*: for every curve there is only one equation, and conversely. The correspondence is also *exact*: the coördinates of every point of the curve satisfy the equation exactly. The present problem is quite different. We shall find that different methods produce different equations for the same set of data. Also, it is obvious that in general an empirical formula is merely an approximation to the true relation between the variables. Frequently a very close correspondence can be found; but if important extraneous factors enter, only a very rough approximation is obtainable. For instance, in (6), varying economic conditions in different years, such as alternating periods of prosperity and depression, might affect the problem materially. Even then, however, the formula might be accurate enough to yield useful results.

126. The method of selected points. Consider an

Example: A set of eight cylindrical bars, all of the same radius and same material but of different lengths, are weighed with results as tabulated * (l in feet, W in pounds). Express W as a function of l .

l	1	2	3	4	5	6	7	8
W	0.38	0.68	1.13	1.47	1.78	2.18	2.60	2.84

* By intention, the data given are quite inaccurate; for if not, in the small-scale drawing to which we are limited in the text, all the points would appear to lie on the line. It is hardly necessary to remark that, in the work of this chapter, the student should make all his drawings on a very generous scale.

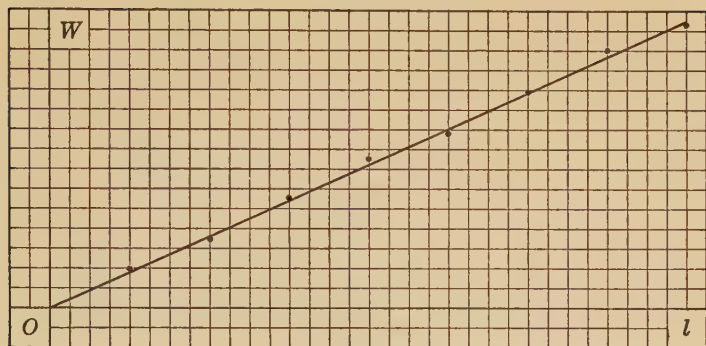


FIG. 104

Plotting the points (1, 0.38), (2, 0.68), etc., with 4 spaces as the horizontal and 5 as the vertical unit, we find that the points follow more or less closely a straight line through the origin. In the present case, of course, we know beforehand that this must be so, since the weight is proportional to the length:

$$(1) \quad W = bl.$$

Now, laying a transparent ruler on the paper, we draw the straight line (through O) which, *on the average*, according to our best judgment, most closely follows the system of points. In 32 horizontal spaces the line rises about 14.3 spaces, which in terms of our scale gives a slope

$$b = \frac{\frac{1}{5}(14.3)}{\frac{1}{4}(32)} = \frac{2.86}{8} = 0.358,$$

and the equation is

$$(2) \quad W = 0.358l.$$

This is known as the *method of selected points*. That is, we select two points on our estimated “best-fitting” line—in the example, the points (0, 0), (8, 2.86)—and find the equation of the line through those points by § 25.

127. Uses for empirical formulas. (a) *Computation.* In the above example, the formula may be used to re-determine the values of W for $l = 1, 2, \dots, 8$, with a probability that the values thus obtained will be more accurate than the observations. We may likewise compute W for any other values of l in the interval $0 < l < 8$.

(b) *Theory.* In the example (§ 126), we know in advance that W is proportional to l ; but frequently no law connecting the variables is known. In such a case we draw a curve through the plotted points. If this curve is approximately a straight line, parabola, or some other familiar curve, we assume an appropriate form for the function—linear, quadratic, etc.—and determine the constants, thus obtaining an *empirical law*. From this law we may be able to develop a satisfactory theory, possibly discovering various properties of the function that would otherwise have been overlooked.

(c) *Applications.* The nature of the applications to be made depends of course on the particular problem in hand. As an application of (2), § 126, we might compute the *density* (weight per unit volume) of the material.

Let w denote the density in pounds per cubic foot. Suppose that the radius is

$$r = 0.213 \text{ in.} = \frac{0.213}{12} \text{ ft.}$$

Then, if V is the volume,

$$W = wV = w\pi r^2 l = 0.358l,$$

$$w = \frac{0.358}{\pi r^2} = \frac{0.358 \times 144}{\pi \times (0.213)^2}$$

$$\begin{aligned} \log w &= \log 0.358 + \log 144 - \log \pi - 2 \log (0.213) \\ &= 2.55833, \quad w = 362 \text{ lbs. per cu. ft.} \end{aligned}$$

128. Residuals. The difference between the observed value of the function and the computed value—i.e. the vertical distance of the plotted point from the curve as drawn—is called the *residual* corresponding to that point. In the example of § 126, for $l = 1$, the observed value of W is 0.38, the formula gives 0.358, the residual is 0.022; for $l = 2$, the respective values are 0.68, 0.716, -0.036 .

129. The method of averages. While the somewhat rough-and-ready method of selected points in many cases gives good enough results for practical purposes, it is open to the serious objection that the choice of the “best-fitting” line depends solely on the judgment of the operator, and hence is to some extent a matter of guess-work. In this chapter, therefore, we shall rely chiefly upon a formal arithmetic process called the *method of averages*. This method gives better results as a rule than the method of selected points—decidedly so when the data are given with great accuracy.

To construct an empirical equation by the method of averages, we *determine the constants so that the algebraic sum * of the residuals is 0*.

130. Linear equation with one constant. In the equation
(1) $y = a + bx,$

assume that a is known, and b is to be determined from a set of observations. The discussion applies without essential change to the case where b is known and a to be found.

Let the given pairs of values be $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)$. At the first point the observed value is y_1 , the value computed by (1) is $a + bx_1$, the residual is $y_1 - a - bx_1$. The succeeding residuals are $y_2 - a - bx_2,$

* By the “method of least squares,” the constants are determined so that *the sum of the squares of the residuals is a minimum*. That method gives still better results, but the computations involved are more laborious.

$y_3 - a - bx_3, \dots, y_n - a - bx_n$. Since by the method of averages the sum of the residuals must be 0, we put

$$(2) \quad y_1 + y_2 + y_3 + \dots + y_n - na \\ - b(x_1 + x_2 + x_3 + \dots + x_n) = 0,$$

and solve for b . This may be formulated as

RULE I: *To determine an empirical equation containing one constant, by the method of averages:*

1. *Substitute the given pairs in the assumed equation.*
2. *Add the equations thus obtained, member by member.*
3. *Solve the resulting equation for the constant.*

It is easily seen that this rule applies not only to the linear equation, but to an equation of any form *containing a single unknown constant*.

Example: Solve the example of § 126 by averages.

Step 1. Substitute the given pairs in * (1), § 126:

$$0.38 = 1b,$$

$$0.68 = 2b,$$

$$1.13 = 3b,$$

$$1.47 = 4b,$$

$$1.78 = 5b,$$

$$2.18 = 6b,$$

$$2.60 = 7b,$$

$$2.84 = 8b.$$

$$\text{Step 2. Add:} \quad \begin{array}{r} 0.38 \\ 0.68 \\ 1.13 \\ 1.47 \\ 1.78 \\ 2.18 \\ 2.60 \\ 2.84 \\ \hline 13.06 \end{array} = \begin{array}{r} 1b \\ 2b \\ 3b \\ 4b \\ 5b \\ 6b \\ 7b \\ 8b \\ \hline 36b \end{array}$$

$$\text{Step 3.} \quad b = \frac{13.06}{36} = 0.363.$$

Thus we find

$$(3) \quad W = 0.363I.$$

* To illustrate the general method, we apply the rule directly. However, it appears from (2) that when $a = 0$ the rule may be simplified slightly, as follows:

1. *Add the given values of x (the independent variable).*
2. *Add the observed values of y (the function).*
3. *Divide the latter by the former to find b .*

131. Comparison of methods. When a formula is obtained by two methods, that formula which produces the smaller *average residual* (apart from sign) is said to represent the better approximation. Computing W by (2), § 126, and by (3), § 130, we obtain (apart from sign) the following residuals respectively, starting with $(0, 0)$:

0.000	0.000
0.022	0.017
0.036	0.046
0.056	0.041
0.038	0.018
0.010	0.035
0.032	0.002
0.094	0.059
0.024	0.064
0.312;	0.282.

Dividing each sum by 9, we get the average residuals 0.0347, 0.0313. Thus the method of averages has produced a slightly better result. However, the chief point is that a second determination by selected points, even by the same operator, might not turn out equally well, whereas the method of averages, being purely numerical, can produce only one result.

EXERCISES

By plotting, verify the existence of an approximate linear relation. Assuming that the first point is exactly on the line, determine the slope (*a*) by selected points; (*b*) by averages. Compute the average residual given by each method.

1.

x	0	1	2	3	4	5	6
y	0	2.3	4.0	7.6	10.3	12.7	14.6

Ans. (*b*) $y = 2.45x$; av. res. 0.336.

2.

x	0	1	2	3	4	5	6
y	0	13	26	35	52	61	76

Ans. (b) $y = 12.52x$; av. res. 1.20.

3.

x	0	2	5	10	20	30	50
y	10	9.6	9.3	8.9	7.6	7.1	4.8

Ans. (b) $y = 10 - 0.109x$; av. res. 0.170.

4.

x	0	5	10	20	30	50	100
y	100	95.7	92.1	81.6	74.0	56.3	10.2

Ans. (b) $y = 100 - 0.884x$; av. res. 0.60.

Solve the following by both methods.

5. An airplane flies at practically constant speed. By landmarks, the pilot estimates his distance (in miles) from the starting point at the end of each hour as shown. Find (a) the speed; (b) the distance after $6\frac{1}{2}$ hrs.

t	1	2	3	4	5	6	7
d	100	205	300	450	550	640	720

Ans. (a) 106 mi. per hr.

6. As a means of computing π , a class of high school students were required to measure the circumferences of a set of circular disks of known radii. The measurements, averaged for the whole class for each disk, were as shown. Find the error in the computed value of π .

r	3	4	5	10
c	18.829	25.092	31.390	62.853

Ans. -0.0015 .

7. It is shown in mechanics that, when a body slides from rest down a smooth inclined plane, the velocity acquired in t seconds is approximately

$$v = g \sin \alpha \cdot t,$$

where g is the *acceleration of gravity* (32.16 ft. per sec. per sec.) and α is the angle of inclination of the plane. For the plane on which the given observations were made, compute b in the formula $v = bt$, and find α .

t	1	2	3	4	5	6	7	8
v	1.31	2.52	3.52	4.72	5.87	7.02	8.17	9.42

Ans. $\alpha = 2^\circ 6'$.

8. In the apparatus called *Atwood's machine*, two weights m_1, m_2 are joined by a cord hung over a pulley. We know from mechanics that, when the system is released, it acquires in time t a velocity

$$v = \frac{m_1 - m_2}{m_1 + m_2} gt.$$

(For the meaning and value of g , see Ex. 7.) In an experiment to determine g , the following values were obtained. (a) Find b in the formula $v = bt$; (b) determine the error in the computed value of g , if the weights were 5.1 and 4.9 lbs.

t	0.5	1.0	1.5	2.0	2.5	3.0	3.5
v	0.35	0.67	0.90	1.26	1.66	1.93	2.26

Ans. (b) + 0.09.

9. A student measures the distance between two points B, C equidistant from the angle of a carpenter's square. Being able to measure correctly to the nearest tenth of an inch, he obtains the results tabulated, for various positions of B and C . Find d in terms of l . What quantity has been computed, and what is the error in its value?

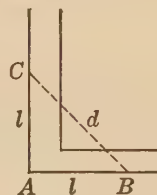


FIG. 105

l	1	2	3	4	5	6	7	8	9	10	11	12
d	1.4	2.8	4.2	5.7	7.1	8.5	9.9	11.3	12.7	14.1	15.6	17.0

Ans. Error - 0.0001.

10. The experiment of Ex. 9 was repeated with $AB = l, AC = 2l$, with results as shown. Answer the same questions.

l	1	2	3	4	5	6	7	8	9	10	11	12
d	2.2	4.5	6.7	8.9	11.2	13.4	15.7	17.9	20.1	22.4	24.6	26.8

Ans. Error + 0.0003.

11. A steel spring 8 in. long is suspended vertically and a weight is attached. The length of the spring for various weights is measured. By Hooke's Law, l is a linear function of W ; determine the function.

W	0.5	2	5	10	15	17.5
l	8.37	9.46	11.68	14.90	18.52	20.02

Ans. $l = 0.70W + 8$.

132. Rectification. When the points do not follow a straight line, so that no linear relation exists, we must try to represent the data by an equation of some other form.

As an example, say we know from theory, or have some other reason to believe, that the variables satisfy the relation

$$(1) \qquad \log y = a + bx^2.$$

Let us introduce two new variables x' , y' such that

$$x' = x^2, \qquad y' = \log y.$$

Equation (1) then becomes

$$y' = a + bx',$$

which is linear. That is, if for each pair of values of x and y we compute the values of x^2 and $\log y$ and plot these values as abscissa and ordinate—briefly, if we plot $(x^2, \log y)$ instead of (x, y) —the points thus plotted must follow a straight line. (If they do not, then the assumed equation (1) cannot be made to fit the data.) This process is called *rectification* * of the curve.

By precisely similar argument, we establish

RULE II: *To rectify the curve*

$$Y = a + bX,$$

where X and Y are any functions whatever of x and y respectively:

1. For each pair of values of x and y given in the table, compute the corresponding values of X and Y .

2. Plot the values thus obtained as abscissa and ordinate respectively—that is, plot the points (X, Y) .

* Elsewhere in mathematics, to *rectify* a curve means to find its *length*, or the length of the arc joining two given points. The two uses of the term must not be confused.

When the curve is “rectified,” two important advantages are gained:

(a) If the points closely follow a straight line, we know definitely that the data can be well represented by an equation of the assumed type.

(b) The method of selected points can then be used.

Frequently, by changing the form of the equation—e.g. by taking the square root or the logarithm of both members (see Exs. 6, 10 below)—we may discover a second transformation by which the curve may be rectified. This possibility should not be overlooked, since the alternative transformation may involve less numerical work than the one directly indicated.

Finally, it should be pointed out that the *straight line* is of no importance except as a means to an end. The only thing of final interest is the equation of the *curve* fitting the data given in the original table.

133. Non-linear equation with one constant. We consider first equations of the form

$$Y = a + bX$$

where either a or b (usually a) is *known*.

Example: When a body slides from rest down a smooth inclined plane (cf. Ex. 7, p. 188), it travels in time t a distance

$$(1) \quad x = \frac{1}{2}g \sin \alpha \cdot t^2.$$

Represent x as a function of t , and find α , for the plane yielding the results given in the first two lines of the table.

t	1	2	3	4	5
x	1.10	4.35	10.05	17.15	26.25
t^2	1	4	9	16	25

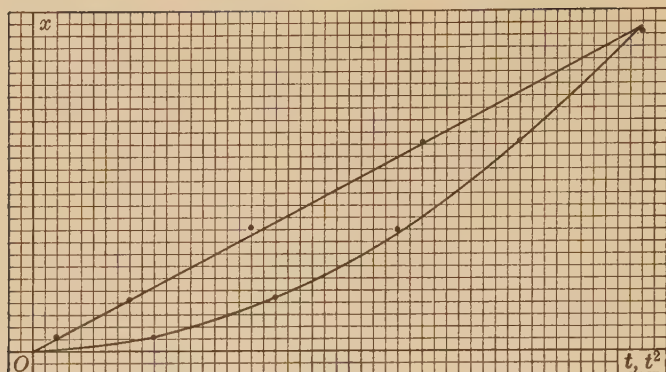


FIG. 106

According to (1), we must expect the data to satisfy an equation of the form

$$(2) \quad x = bt^2.$$

This curve will evidently be rectified if we plot (t^2, x) . Thus the first step is to compute the values of t^2 as shown in the third line. Plotting t^2 as abscissa with 2 spaces as the unit, and x as ordinate with 1 space as the unit, we find that the points follow a straight line quite closely. The student should plot the points on a larger scale and determine the slope b of the estimated "best-fitting" line as in § 126. By the method of averages (Rule I, § 130, or footnote, p. 186) we find

$$58.90 = 55b, \quad b = 1.071, \quad x = 1.071t^2.$$

Computing values of x from this equation, we draw the curve, with 10 spaces as the t -unit and 1 as the x -unit. The points given in the table are shown by dots. Finally,

$$\frac{1}{2}g \sin \alpha = b = \frac{58.90}{55}, \quad \sin \alpha = \frac{2 \times 58.90}{55 \times 32.16},$$

$$\log \sin \alpha = 8.82347 - 10, \quad \alpha = 3^\circ 49'.$$

EXERCISES

1. Solve the example by plotting $\left(t, \frac{x}{t}\right)$.
2. Solve the example by plotting $(\log t, \log x)$.
3. Verify by rectification that the data may be fitted by an equation of the form $y = bx^3$, determine the constant, and plot the curve.

x	0.6	0.8	1.0	1.2	1.4
y	0.161	0.374	0.806	1.302	2.150

Ans. $y = 0.773x^3$.

4. Discover by plotting that the data suggest an equilateral hyperbola, verify by rectification, and determine the constant.

x	0.5	0.8	1.0	1.5	2.0
y	7.11	4.23	3.69	2.20	1.64

5. It is shown in mechanics that the kinetic energy of a body of mass m moving with velocity v is

$$E = \frac{1}{2}mv^2.$$

Find a value of m to fit the following data, and plot the curve.

v	0.8	1.2	1.5	1.7	2.2
E	2.17	4.92	7.95	10.22	17.80

Ans. 7.14.

6. Solve Ex. 5 by plotting $(\log v, \log E)$.

7. In Ex. 6, p. 188, the disks were made of sheet metal weighing 11.03 oz. per sq. ft. The disks were weighed and the results averaged over the whole class, as follows (r in inches, W in ounces). What was the error in the computed value of π ?

r	3	4	5	10
W	2.167	3.853	6.022	24.080

Ans. + 0.0023.

8. In Ex. 7, what density would produce the value of π correct to 4 decimal places? (Solve the problem again, with the same values of r and W , taking $\pi = 3.1416$ and considering the density factor as the unknown constant.)

Ans. 11.038.

9. In the inclined-plane experiment (Ex. 7, p. 188) the body, starting from rest, acquires at a distance x from the starting point a velocity given by the formula

$$v^2 = 2g \sin \alpha \cdot x.$$

For the following experiment, rectify by plotting (x, v^2) , find v in terms of x , plot the curve, and determine α .

x	2	4	6	8	10
v	3.4	4.8	5.9	6.9	7.8

Ans. $v^2 = 5.93x; \alpha = 5^\circ 17'.$

10. Solve Ex. 9 by using the coördinates $(\sqrt{x}, v)$.

Ans. $v^2 = 5.90x; \alpha = 5^\circ 16'.$

11. In the theory of fluid pressure it is shown that when a rectangular plate of dimensions b, h is submerged vertically with an edge b in the surface, the force on one face of the plate is

$$F = \frac{1}{2} w b h^2,$$

where w is the density of the fluid. A liquid was allowed to run into a rectangular trough of width 18 in. and the force on one end measured for various depths h (in feet). Express F as a function of h , plot the curve, and find the density of the liquid.

h	0.25	0.5	0.75	1.0	1.25	1.5
F	2.9	12.3	25.8	47.5	73.4	104.2

Ans. $F = 46.8h^2; w = 62.4$ lbs. per cu. ft.

12. A number of right circular cylinders closed at both ends, of altitude 10 in. but different radii, are constructed of thin sheet metal weighing 2.737 oz. per sq. ft. As an experiment to determine π , the cylinders are weighed with results as shown (r in inches, W in ounces). Find W in terms of r , plot the curve, and determine the error in the value of π .

r	1.0	1.5	3.0	3.5	4.0
W	1.332	2.052	4.652	5.662	6.683

Ans. Error $+ 0.003$.

13. Discover by plotting that the data suggest an equilateral hyperbola, verify by rectification, and find the equation.

x	0.20	0.40	0.81	1.15	1.45
y	0.63	0.73	1.01	1.31	1.57

14. It is known from physics that the attraction between two magnetic poles is inversely proportional to the square of the distance between them:

$A = \frac{k}{d^2}$. Find k for the following experiment, and plot the curve.

d	0.5	0.6	0.75	1.0	1.5	2.0
A	0.0352	0.0243	0.0162	0.0087	0.0041	0.0022

15. Solve Ex. 14 by plotting ($\log d$, $\log A$).

16. A number of hollow spheres all have the same inner radius r but different outer radii R . The volume of material in each sphere is found by weighing to be as shown. Find V in terms of R ; plot the curve; find r .

R	3	4	5	6
V	60.2	216.0	463.8	857.7

Ans. $r = 2.33$.

17. As an experiment to determine g (cf. the example of § 133), t was measured at $x = 25$, for various values of α as tabulated. Find the error in the value of g , and plot the (α, t) -curve.

α	1°	1° 30'	2°	2° 30'	4°	6°	9°
t	9.46	7.74	6.72	5.99	4.77	3.93	3.21

18. Plot the curve $y^2 = x$ accurately in the interval $0 < x < 4$. By counting squares, determine the area bounded by the curve, the x -axis, and the line $x = h$ for $h = 1, 2, 3, 4$. Show by rectification that the data fit a formula $A = kh^{3/2}$, and find k to four decimal places.

19. Plot the curve $y = 2^x$ accurately in the interval $0 < x < 4$. By counting squares, determine the area bounded by the curve, the axes, and the line $x = h$, for $h = 1, 2, 3, 4$. Show that the data fit a formula $A = k(2^h - 1)$, and find k to four decimal places.

20. Plot the curve $xy = 1$ accurately in the interval $1 < x < 5$. By counting squares, determine the area bounded by the curve, the x -axis, the line $x = 1$, and the line $x = h$, for $h = 2, 3, 4, 5$. Show that the data fit a formula $A = k \log_{10} h$, and find k to four decimal places.

21. Find a pair of coördinates to rectify the curve

$$\frac{y+b}{y-b} = \frac{2x+2b}{x-2b} \quad \text{Ans. } (3x - 4y, xy).$$

134. Equations containing two constants. Consider such equations as

$$y = a + bx, \quad x = a + bt^2, \quad S = ar^2 + br,$$

where both a and b are unknown. If we were to apply Rule I (§ 130) in such a case, we should finish up with one equation in two unknowns—a true equation, but insufficient to determine the unknowns. To overcome this difficulty, we divide the equations into two groups and apply Rule I to each group separately:

RULE III: *To determine an empirical formula containing two constants, by the method of averages:*

1. *Substitute the given pairs of values in the assumed formula, dividing the equations thus obtained into two groups.*
2. *Add the equations, member by member, in each group.*
3. *Solve the resulting two equations for the constants.*

Usually the best result is obtained if the two groups are of equal size or nearly so.

By an obvious extension, when the formula involves three constants we divide the equations into three groups; etc.

Example: Determine a linear equation $y = a + bx$ to fit the given data.

x	1	2	3	4	5	6
y	11.6	11.1	10.1	9.6	9.0	8.1

The student should plot the points on a large scale, verifying the existence of an approximate linear relation, and determine the equation of the best-fitting line by selected points (note that since no one point is known to lie exactly on the line, the ruler may be placed in any position). By the method of averages:

Step 1. Divide the data into two equal groups, and substitute in $y = a + bx$:

$$\begin{array}{rcl} 11.6 & = & a + b \\ 11.1 & = & a + 2b \\ 10.1 & = & a + 3b \end{array} \qquad \begin{array}{rcl} 9.6 & = & a + 4b \\ 9.0 & = & a + 5b \\ 8.1 & = & a + 6b \end{array}$$

Step 2. Add: $32.8 = 3a + 6b, \quad 26.7 = 3a + 15b.$

Step 3. Solve for a and b : $a = 12.29, b = -0.678.$

EXERCISES

1. Fit a linear equation to the given data (a) by selected points; (b) by averages, grouping the first four and the last three pairs. (c) Compute the average residual.

x	1	2	3	4	5	6	7
y	1.82	4.19	6.90	9.21	11.65	14.36	16.72

Ans. (b) $y = 2.49x - 0.69$; (c) 0.067.

2. Solve Ex. 1, grouping the first three and the last four pairs.

Ans. (b) $y = 2.48x - 0.66$; (c) 0.066.

3. The force necessary to lift a weight by means of a pulley was measured as shown. Find F in terms of W , and tabulate the values of F at intervals of 5 lbs. from $W = 5$ to $W = 30$.

W	5	8	12	20	25	30
F	1.65	2.15	2.65	4.20	5.10	5.90

4. For the lever of Fig. 107, with the fulcrum at A , it is shown in mechanics that the force F necessary to balance the weight W is

$$F = \frac{l_1}{l_2} W + \frac{w}{2l_2}(l_1^2 - l_2^2),$$

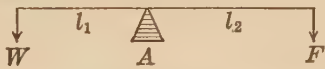


FIG. 107

where w is the weight of the lever per linear unit. The following data were obtained with a lever weighing 4 lbs. per ft. Verify that F is a linear function of W ; determine the coefficients; find l_1, l_2 .

W	15	20	30	50	75	100
F	1.6	3.9	9.0	18.8	31.3	44.2

5. For the lever of Fig. 108, with the fulcrum at A , it is shown in mechanics that the force necessary to balance the weight W is

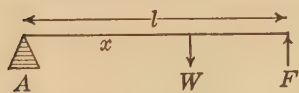


FIG. 108

$$F = \frac{W}{l} \cdot x + \frac{1}{2}wl,$$

where w is the weight of the lever per linear foot. The following data were obtained by placing a weight of 10 lbs. at various distances x from A . Find w and l .

x	0	2	4	8	12	15
F	41.1	42.6	43.7	46.2	48.8	50.5

Ans. $w = 5$ lbs. 2 oz. per ft.; $l = 16.0$ ft.

6. The work done in changing the velocity of a moving body is

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2,$$

where m is the mass of the body, v_0 the initial velocity, and v the final velocity. For the data given, verify by rectification that there exists a relation $W = a + bv^2$; determine the constants and plot the curve; find m and v_0 .

v	8	10	12	14	16	18	20
W	26	115	227	358	515	684	880

Ans. $m = 5.10$; $v_0 = 7.41$.

7. Find a pair of coördinates to rectify the curve $y = \frac{x}{a + bx}$.

8. Find a pair of coördinates to rectify the curve $y = \frac{a + x}{a + bx}$.

9. When a projectile is thrown with an initial velocity v_0 inclined at an angle α to the horizontal, the equation of its path (with all resistances neglected) is

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2} \sec^2 \alpha.$$

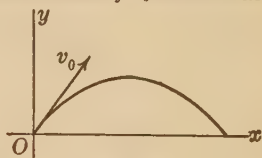


FIG. 109

For the given data, verify the formula $y = ax + bx^2$, find the coefficients, and plot the path; find α and v_0 , and the highest point (vertex of the parabola).

x	2	4	6	8	10	12	14	16	18
y	3.8	6.8	8.9	10.8	11.7	12.1	11.8	10.8	9.1

Ans. $y = 2.04x - 0.086x^2$; $\alpha = 63.9^\circ$, $v_0 = 9.5$ ft. per sec.; $V: (11.9, 12.1)$.

10. The given data represent the power P developed at one end of a transmission line and the power Q delivered at the other end. Verify the formula

$$Q = aP + bP^3$$

by rectification, determine the coefficients, and plot the curve.

P	100	200	300	400	500
Q	95	185	270	350	415

$$\text{Ans. } Q = 0.95P - 0.0000005P^3.$$

11. A *cantilever beam* is a beam rigidly fixed horizontally at one end and otherwise unsupported (e.g. a springboard). When the beam bears a load at the free end and the weight of the beam itself is negligible, the *deflection* y at a distance x from the fixed end is

$$y = k(3lx^2 - x^3),$$

where l is the length of the beam and k is a constant depending on the magnitude of the load and on the physical characteristics of the beam. For the data given (x in feet, y in inches), verify that $y = ax^2 + bx^3$; plot the curve; find l .

x	3	5	7	9	11	12
y	0.59	1.56	2.83	4.36	6.05	6.90

$$\text{Ans. } y = 0.072x^2 - 0.002x^3; l = 12 \text{ ft.}$$

12. A projectile (see Ex. 9) is thrown against a wall distant h feet away (horizontally) and the height

$$y = h \tan \alpha - \frac{gh^2}{2v_0^2} \sec^2 \alpha$$

at which it strikes is measured. With constant v_0 and varying α the following data were obtained. (a) Show that the curve

$$y = a \tan \alpha + b \sec^2 \alpha$$

may be rectified by plotting $(\sin 2\alpha, y \cos^2 \alpha)$; (b) determine the coefficients and plot the (α, y) -curve, interpolating for $\alpha = 37\frac{1}{2}^\circ, 52\frac{1}{2}^\circ, 67\frac{1}{2}^\circ$; (c) find h and v_0 .

α	15°	$22\frac{1}{2}^\circ$	30°	45°	60°	75°
y	0.51	1.84	3.02	5.94	9.32	7.40

$$\text{Ans. (b) } y = 10.22 \tan \alpha - 2.10 \sec^2 \alpha.$$

13. Solve Ex. 12 by plotting $(\csc 2\alpha, y \cot \alpha)$.

135. The power function: parabolic and hyperbolic curves. Two functional forms of very frequent occurrence are *

$$(1) \quad y = ax^b, \quad (b > 0)$$

$$(2) \quad y = \frac{a}{x^b}. \quad (b > 0)$$

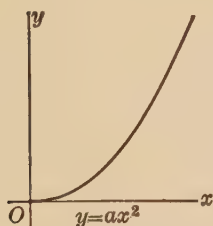


FIG. 110

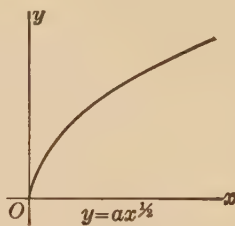


FIG. 111

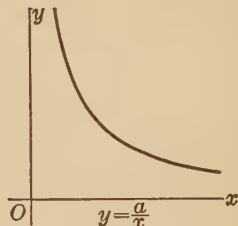


FIG. 112

I: Parabolic curves. When $b = 2$, equation (1) represents the parabola

$$(3) \quad y = ax^2;$$

when $b = \frac{1}{2}$, the parabola

$$(4) \quad y = ax^{1/2}, \quad \text{i.e.} \quad y^2 = a^2x.$$

For all other values of $b > 1$, and for $x > 0$, the curves (1) resemble in a general way the parabola (3); for $0 < b < 1$ they all resemble (4). For this reason the curves (1) are called † *parabolic curves*. All the parabolic curves pass through $(0, 0)$, $(1, a)$.

II: Hyperbolic curves. When $b = 1$, equation (2) represents the equilateral hyperbola

$$(5) \quad y = \frac{a}{x}.$$

* Evidently, if b is not restricted to positive values, the notation $y = ax^b$ includes both (1) and (2). On account of the wide difference in the curves, it makes for clearer understanding to distinguish the two forms explicitly.

† The curve $y = x^3$ is sometimes called a *cubical parabola*; $y = x^{3/2}$, a *semi-cubical parabola*; etc. It is hardly necessary to say that none of these curves are parabolas in the strict sense (§ 47).

All other curves of the family (2) resemble this hyperbola for positive values of x , and are called *hyperbolic curves*. They all pass through $(1, a)$, and all have the coördinate axes as asymptotes.

To rectify either (1) or (2) we merely take the logarithm of each member: *

$$(6) \quad \log y = \log a + b \log x,$$

$$(7) \quad \log y = \log a - b \log x.$$

Whenever the curve of a given set of data (functional form unknown) appears to resemble any one of the curves in Figs. 110–112, it is worth while to test by plotting $(\log x, \log y)$.

Example: Fit an empirical equation to the data given in the first two lines.

x	1	2	3	5	8	10
y	1.751	1.202	0.967	0.761	0.607	0.528
$\log x$	0	0.30103	0.47712	0.69897	0.90309	1.00000
$\log y$	0.24329	0.07990	9.98543	9.88138	9.78319	9.72263

Plotting the points (the upper group in Fig. 113), with 4 spaces as the unit in each direction, we see that the curve resembles to some extent the hyperbola of Fig. 112. We therefore make the entries shown in the last two lines of the table and plot $(\log x, \log y)$, the lower group in Fig. 113, with 40 spaces as the unit for $\log x$ and 10 for $\log y$. Since these points are nearly in a straight line, the assumption of (2) is justified and we substitute in (7):

* These curves appear so often that specially ruled paper, called *logarithmic paper*, has been constructed and is obtainable on the market. When this paper is used the curve is rectified by plotting the given data directly, without recourse to (6) or (7).

$$\begin{aligned}
 0.24329 &= \log a \\
 0.07990 &= \log a - 0.30103b \\
 9.98543 - 10 &= \log a - 0.47712b \\
 \hline
 0.30862 &= 3 \log a - 0.77815b, \\
 9.88138 - 10 &= \log a - 0.69897b \\
 9.78319 - 10 &= \log a - 0.90309b \\
 9.72263 - 10 &= \log a - 1.00000b \\
 \hline
 9.38720 - 10 &= 3 \log a - 2.60206b.
 \end{aligned}$$

These equations give $a = 1.714$, $b = 0.505$, so that

$$y = \frac{1.714}{x^{0.505}}.$$

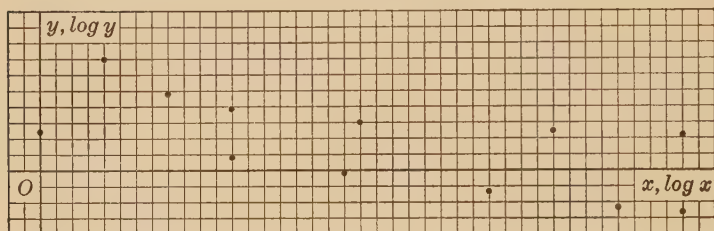


FIG. 113

EXERCISES

Discover a trial formula by plotting the points; verify by rectification; determine the coefficients and plot the curve.

1.

x	3	5	8	10	15	20
y	2.81	4.05	5.46	6.54	8.54	10.32

2.

x	1	2	3	4	5	6
y	2.32	5.30	8.62	12.17	15.93	19.87

3.

x	1	2	3	4	5	6	8
y	1.78	1.17	0.91	0.77	0.67	0.58	0.50

4. The time t required for a body to fall a distance x was observed as shown (x in feet, t in seconds). Find t in terms of x .

x	2	4	6	8	12	16
t	0.36	0.52	0.64	0.75	0.91	1.04

$$\text{Ans. } t = 0.254x^{0.513}.$$

5. In 1930 the world's free-style swimming records were as given (x in meters). Determine t as a function of x .

x	100	200	300	400	500	1000
t	57.4	2:08.0	3:33.5	4:50.3	6:08.4	13:04.2

$$\text{Ans. } t = 0.317x^{1.135}.$$

6. In 1927 the world's free-style swimming records, 20-yard pool, were as given (x in yards). Express t as a function of x .

x	50	100	150	220	300	500
t	22.8	49.8	1:25.0	2:08.6	3:12.4	5:28.4

$$\text{Ans. } t = 0.233x^{1.171}.$$

7. In Ex. 6, the average speed of the swimmer, as computed from the answer to that exercise, is

$$v_a = \frac{x}{t} = \frac{1}{0.233x^{0.171}} = \frac{4.298}{x^{0.171}}.$$

Check this answer by computing v_a independently from the following table.

x	50	100	150	220	300	500
v_a	2.193	2.006	1.765	1.711	1.559	1.523

8. In 1936 the world's records in track events were as shown (x in miles, t in seconds). Find t in terms of x .

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	5
t	20.6	46.4	109.8	246.8	538.4	1446.2

SOLID ANALYTIC GEOMETRY

CHAPTER XV

COÖRDINATES IN SPACE

136. Rectangular cartesian coördinates. To determine the position of a point in three-dimensional space, it is obvious that three magnitudes must be given. Thus to locate a point in the interior of a room, we may give its height above the floor and its distances from two adjacent walls; the position of a point in the interior of the earth may be fixed by its depth below the surface together with the latitude and longitude of the point on the surface above it.

Usually the simplest and most convenient method of fixing the position of a point in space is *by means of its distances from three mutually perpendicular planes*, as in the first example just cited. These distances are the *rectangular cartesian coördinates* of the point; the three planes are the *coördinate planes*, their three lines of intersection are the *coördinate axes*, and their point of intersection is the *origin*. The coördinates are denoted by the letters x, y, z , and are written $P : (x, y, z)$, or merely (x, y, z) . The three axes are called the x -axis, the y -axis, and the z -axis; the three planes are the xy -plane (containing the x - and y -axes), the yz -plane, and the zx -plane.

Evidently a definite positive sense must be chosen for each coördinate; that is, the coördinates are *directed segments*, as in plane geometry.

Space is divided by the coördinate planes into eight

compartments, or *octants*. The region in which all three coördinates are positive is called the *first octant*; there will be no occasion to refer to the others by number.

137. Figures. Plane pictures representing figures in space may be drawn according to various methods. The system customarily used in analytic geometry, and the one adopted in this book, is a form of “oblique parallel projection.” In this system, *parallel lines are represented by parallel lines*: i.e. if two lines in space are parallel, they are shown in the figure by lines that are actually parallel, instead of by lines that run to a “vanishing point,” as in ordinary perspective drawings.

Two of the axes are represented by perpendicular lines, while the third, which of course is perpendicular to the other two, is shown by a line drawn in any convenient direction. The axes are usually placed as in Fig. 114, the positive half of each axis being the part drawn in full. Of course a different arrangement is permissible, and should be adopted when a clearer or better figure is obtained by so doing.

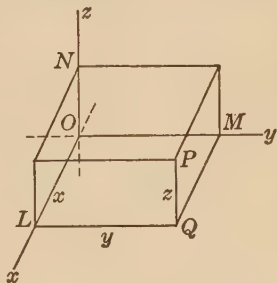


FIG. 114

With the axes arranged as shown, figures in the yz -plane, or in a plane parallel to that plane, are drawn in their true form and proportions, but all others are distorted, due to foreshortening in the direction of x .

In Fig. 114, the coördinates of P are $OL = x$, $OM = y$, $ON = z$. To plot the point, we lay off $OL = x$; then, since parallels are represented by parallels, draw $LQ = y$ parallel to Oy , then $QP = z$ parallel to Oz , thus arriving at P .

138. Distance between two points. To find the distance d between any two points $P_1 : (x_1, y_1, z_1)$ and $P_2 : (x_2, y_2, z_2)$, we note that, in Fig. 115,

$$d = \sqrt{P_1M^2 + MP_2^2} = \sqrt{P_1L^2 + LM^2 + MP_2^2}.$$

Now

$$P_1L = IJ = GH = x_2 - x_1;$$

similarly,

$$LM = y_2 - y_1, \quad MP_2 = z_2 - z_1.$$

Thus

$$(1) \quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

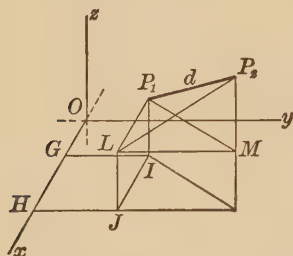


FIG. 115

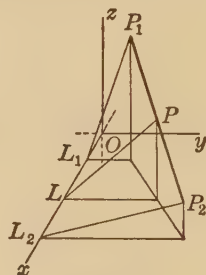


FIG. 116

139. Midpoint of a line segment. To determine the point $P : (x, y, z)$ bisecting the line segment joining $P_1 : (x_1, y_1, z_1)$ and $P_2 : (x_2, y_2, z_2)$, let us drop perpendiculars PL, P_1L_1, P_2L_2 from these points to Ox . Then, in Fig. 116,

$$OL = OL_1 + \frac{1}{2}L_1L_2;$$

that is,

$$x = x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_1 + x_2).$$

In this way we obtain the following formulas:*

$$(1) \quad \begin{cases} x = \frac{1}{2}(x_1 + x_2), \\ y = \frac{1}{2}(y_1 + y_2), \\ z = \frac{1}{2}(z_1 + z_2). \end{cases}$$

* Cf. § 7. General formulas analogous to those of § 6 are easily obtained if desired.

EXERCISES

1. Plot the points $(2, 4, 1)$, $(3, -1, 2)$, $(-2, 2, -3)$, $(0, 1, -1)$.
 2. Plot the points $(3, 2, 2)$, $(-2, 4, 1)$, $(3, -1, -3)$, $(2, -3, 0)$.
 3. Through the point $(2, 3, 4)$ draw lines intersecting each of the coördinate axes at right angles.
 4. In each coördinate plane, draw a line through O making an angle of 45° with each of the axes, and a perpendicular to this line through an arbitrary point in that plane.
 5. Draw a rectangular parallelepiped, or *box*, with its edges parallel to the axes and having the points $(0, 0, 0)$, $(3, 4, 5)$ as ends of a diagonal.
 6. In each coördinate plane, draw a circle of radius a with center at O .
 7. In the zx -plane, plot the parabola $z^2 = 4ax$.
 8. In the xy -plane, plot the parabola $x^2 = 4ay$.
 9. What is the distance of the point (x, y, z) from Ox ? From Oy ? From Oz ? From O ?
 10. Where is a point situated if

(a) $x = 0$?	(b) $z = 0$?	(c) $x = y = 0$?
(d) $y = z = 0$?	(e) $x = 2$?	(f) $x = 2, y = 1$?
(g) $x = z$?	(h) $y = z, x = 0$?	(i) $x = y = z$?
- Find the distances between the following pairs of points.
11. $(3, -1, 2)$, $(5, -2, -3)$.
 12. $(1, 7, -4)$, $(-3, -2, -2)$.
 13. $(\frac{3}{4}, 0, -\frac{3}{4})$, $(\frac{1}{4}, 2, -\frac{3}{4})$.
 14. $(1, -2, -\frac{3}{4})$, $(-\frac{1}{4}, 0, \frac{3}{4})$.
 15. (a) Prove that the points $(6, 9, 10)$, $(1, -1, -5)$, $(6, -11, 0)$ form a right triangle; (b) find its area. Ans. (b) $25\sqrt{21}$.
 16. (a) Prove that the points $(3, 5, 1)$, $(2, 3, -2)$, $(6, 1, -2)$ form a right triangle; (b) find its area. Ans. (b) $\sqrt{70}$.
 17. (a) Prove that the points $(6, 2, 4)$, $(2, 0, -2)$, $(4, -2, 10)$ form an isosceles triangle; (b) find its area. Ans. (b) $6\sqrt{19}$.
 18. (a) Prove that the points $(6, -2, 3)$, $(1, 3, 2)$, $(0, 2, -5)$ form an isosceles triangle; (b) find its area. Ans. (b) $\sqrt{638}$.
 19. Prove that the points $(6, 1, -3)$, $(0, -2, 3)$, $(10, 3, -7)$ lie in a straight line.
 20. Prove that the points $(1, -1, 3)$, $(3, 2, 1)$, $(-3, -7, 7)$ lie in a straight line.
 21. Do the points $(1, -1, -3)$, $(9, 11, -11)$, $(-4, -9, 2)$ lie in a straight line?
 22. Do the points $(13, 10, -7)$, $(-3, -10, 11)$, $(1, -5, 7)$ lie in a straight line?

Find the midpoint of the segment joining the given points.

23. $(3, -5, 4)$, $(6, -3, -1)$. 24. $(0, 3, -2)$, $(3, 5, -6)$.

25. $(-\frac{1}{4}, \frac{3}{8}, -\frac{2}{3})$, $(\frac{5}{8}, \frac{1}{2}, -\frac{2}{3})$. 26. $(\frac{5}{8}, -\frac{3}{8}, -\frac{1}{4})$, $(3, \frac{7}{4}, -\frac{1}{2})$.

27. Solve Ex. 15 (a) another way. 28. Solve Ex. 16(a) another way.

29. The line from $(1, 4, -3)$ to $(3, -1, 2)$ is doubled. Find the endpoint.

30. The line from $(4, -1, 2)$ to $(3, -3, 5)$ is trebled. Find the endpoint. *Ans.* $(1, -7, 11)$.

31. The line from $(0, 3, 5)$ to $(2, -2, 4)$ is quadrupled. Find the endpoint. *Ans.* $(8, -17, 1)$.

32. The line from $(3, 5, -3)$ to $(5, 9, -7)$ is extended by half its own length. Find the endpoint. *Ans.* $(6, 11, -9)$.

33. Do the points $(3, 5, 1)$, $(2, 4, 6)$, $(3, -1, 7)$, $(0, 2, 4)$ form a parallelogram? *Ans.* No.

34. Do the points $(3, -3, 5)$, $(1, 3, 4)$, $(-5, 4, 6)$, $(-1, 1, 2)$ form a parallelogram? *Ans.* No.

35. Prove that the points $(3, 3, 3)$, $(1, 2, -1)$, $(4, 1, 1)$, $(6, 2, 5)$ form a parallelogram.

36. Prove that the points $(4, 1, 2)$, $(2, 1, 1)$, $(3, 3, -1)$, $(5, 3, 0)$ form a rectangle.

37. A point is at the distance $\sqrt{33}$ from Ox , $2\sqrt{7}$ from Oy , $\sqrt{11}$ from Oz . Find its coördinates. *Ans.* $(\pm\sqrt{3}, \pm 2\sqrt{2}, \pm 5)$.

38. A moving point is always equidistant from the points $(-2, 3, -1)$, $(1, -2, 0)$. Find the equation of its locus. *Ans.* $6x - 10y + 2z + 9 = 0$.

39. A moving point is always at the distance 3 from the point $(1, -2, 2)$. Find the equation of its locus.

$$\text{Ans. } x^2 + y^2 + z^2 - 2x + 4y - 4z = 0.$$

40. Prove that the straight lines joining the midpoints of opposite sides of any quadrilateral (not necessarily plane) intersect and bisect each other.

140. Direction angles; direction cosines. Given any directed line λ passing through the origin, the angles α , β , γ formed by this line with the x -, y -, and z -axes are called the *direction angles* of the line, and the cosines of these angles are the *direction cosines*. Direction cosines are denoted by l , m , n respectively:

$$l = \cos \alpha, \quad m = \cos \beta, \quad n = \cos \gamma.$$

More generally, if the given line does not pass through the origin, its direction angles and direction cosines are defined as *equal to those of the parallel line through the origin*.

Given the three direction cosines l, m, n of any directed line λ' , let $P : (x, y, z)$ be any point on the parallel λ through the origin, and denote the distance OP by ρ . Evidently

$$x = OP \cos \alpha = l\rho;$$

similarly,

$$y = m\rho, \quad z = n\rho.$$

Thus the coördinates of P can be found if l, m, n are known, so that the line λ is determined. It follows that *the direction of any line is determined if its direction cosines are given*.

If the positive sense on the line be reversed, its direction angles are replaced by their supplements, and the signs of the direction cosines are changed. Hence the direction cosines of an undirected line are ambiguous in sign.

In Fig. 117, we have

$$x^2 + y^2 + z^2 = \rho^2,$$

or

$$l^2\rho^2 + m^2\rho^2 + n^2\rho^2 = \rho^2.$$

This gives the very important result:

The direction cosines of any line satisfy the relation

$$(1) \quad l^2 + m^2 + n^2 = 1.$$

It should also be pointed out that, given any set of numbers l, m, n satisfying (1), there will always be a line having those numbers as direction cosines; in fact, the line in question is obviously the line through $(0, 0, 0), (l, m, n)$.

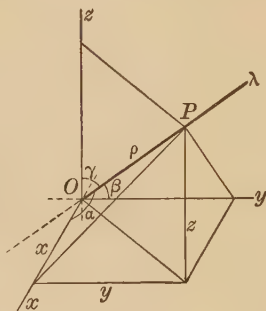


FIG. 117

141. Direction components. If the direction cosines of a line are proportional to three numbers a, b, c , the actual values of the cosines must be

$$l = ka, \quad m = kb, \quad n = kc,$$

where k is a quantity as yet undetermined. Substituting these values of l, m , and n in (1), § 140, we get

$$k^2(a^2 + b^2 + c^2) = 1, \text{ or } k = \frac{1}{\sqrt{a^2 + b^2 + c^2}}.$$

Substituting this value of k in the equations above, we find:

If the direction cosines of a line are proportional to any three numbers a, b, c , their actual values are

$$(1) \quad l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad m = \frac{b}{\sqrt{a^2 + b^2 + c^2}},$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

Any set of numbers a, b, c proportional to the direction cosines of a line are called *direction components* for that line.

The ambiguity of sign mentioned in § 140 appears here in the fact that we might equally well choose the negative sign before the radical.

142. Radius vector of a point. The directed segment OP from the origin to the point $P : (x, y, z)$ is called the *radius vector* of P ; its length is

$$(1) \quad \rho = \sqrt{x^2 + y^2 + z^2}.$$

By § 140, the coördinates of the point in terms of the radius vector and its direction cosines are

$$(2) \quad x = l\rho, \quad y = m\rho, \quad z = n\rho.$$

Hence *the direction cosines of the radius vector of a point are proportional to the coördinates of the point.*

143. Direction cosines of the line through two points.

Let d be the distance between the points $P_1 : (x_1, y_1, z_1)$, $P_2 : (x_2, y_2, z_2)$. Then the direction cosines of the line P_1P_2 are

$$l = \frac{P_1L}{d} = \frac{x_2 - x_1}{d},$$

$$m = \frac{P_1M}{d} = \frac{y_2 - y_1}{d},$$

$$n = \frac{P_1N}{d} = \frac{z_2 - z_1}{d}.$$

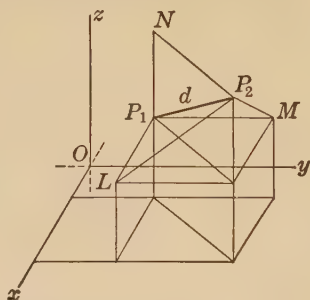


FIG. 118

Hence:

The direction cosines of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

EXERCISES

1. A line makes an angle of 45° with Ox and 60° with Oz . What angle does it make with Oy ?

2. A line has the direction cosines $l = \frac{3}{10}, m = \frac{2}{5}$. What angle does it make with Oz ? Ans. 30° .

3. A line has direction cosines $l = \frac{7}{10}, n = \frac{1}{10}$. What angle does it make with Oy ? Ans. 45° .

4. A line has direction components 3, 4, 5. What angle does it make with Oz ? Ans. 45° .

Draw the following lines.

5. Through O , with direction components 3, 1, 2.

6. Through O , with direction components 2, 5, 4.

7. Through (3, 5, 1), with direction components 2, -3, -1.

8. Through (6, 2, 4), with direction components -3, 2, -2.

9. For each of the following points, find the length and direction cosines of the radius vector: (a) (2, 3, 3); (b) (-3, 5, -2); (c) (2, 1, 0).

10. Where must a point lie if its radius vector has (a) $l = 0$? (b) $l = m = 0$? (c) $l = \frac{1}{2}$? (d) $l = 1$? (e) $l = n = \frac{1}{2}\sqrt{2}$?

11. Can a line be drawn at an angle of 30° with Ox and 45° with Oz ?

12. Prove that it is impossible to draw a line two of whose direction angles shall be less than 45° .

13. A point is at the distance 5 from O , and its radius vector makes an angle of 60° with Oy and 45° with Oz . Find its coördinates.

14. A point is at the distance 3 from O , and its radius vector has $l = \frac{1}{3}$, $m = \frac{5}{6}$. Find its coördinates. *Ans.* $(1, \frac{5}{2}, \pm \frac{1}{2}\sqrt{7})$.

15. A point is at the distance 28 from O , and its radius vector has direction components 2, -3, 6. Find its coördinates.

Ans. $(\pm 8, \mp 12, \pm 24)$.

16. A point is at the distance 18 from O , and its radius vector has direction components 8, 1, -4. Find its coördinates. *Ans.* $(\pm 16, \pm 2, \mp 8)$.

17. A point is at the distance $2\sqrt{5}$ from Oy , 8 from O , and its radius vector makes an angle $\frac{1}{3}\pi$ with Ox . Find its coördinates.

18. A point is at the distance $2\sqrt{3}$ from Oz , 4 from O , and its radius vector has $l = \frac{3}{4}$. Find its coördinates. *Ans.* $(3, \pm \sqrt{3}, \pm 2)$.

19. A point is at the distance 4 from the yz -plane, and its radius vector has $m = \frac{3}{4}$, $n = \frac{1}{4}\sqrt{6}$. Find the point. *Ans.* $(4, 12, 4\sqrt{6})$.

20. A point is at the distance 2 from the yz -plane, and its radius vector has $\alpha = 60^\circ$, $\beta = 45^\circ$. Find its coördinates. *Ans.* $(2, 2\sqrt{2}, \pm 2)$.

21. A point lies on a sphere of radius 5 described about the origin. If it is at a distance 4 from the xy -plane and its radius vector makes an angle of $\frac{1}{3}\pi$ with Ox , find its coördinates. *Ans.* $(\frac{5}{2}, \pm \frac{1}{2}\sqrt{11}, 4)$.

22. A point is at distance 5 from O , 4 from the yz -plane, and its radius vector has $n = \frac{1}{5}$. Find the point. *Ans.* $(4, \pm 2\sqrt{2}, 1)$.

23. A point is distant 1 from the yz -plane, 2 from the zx -plane, and its radius vector has $n = \frac{1}{3}$. Find the point. *Ans.* $(1, 2, \frac{1}{3}\sqrt{10})$.

24. A point is distant 3 from the yz -plane, 1 from the xy -plane, and its radius vector has $m = \frac{2}{3}$. Find the point. *Ans.* $(3, 2\sqrt{2}, 1)$.

25. A point is distant 4 $\sqrt{10}$ from Ox , and its radius vector has direction components 3, 2, 6. Find the point. *Ans.* $(\pm 6, \pm 4, \pm 12)$.

26. A point is at the distance $\sqrt{5}$ from the z -axis, 1 from the xy -plane, and its radius vector makes an angle $\frac{1}{4}\pi$ with Ox . Find its coördinates.

27. A point is at the distance $\sqrt{2}$ from Oz , $\sqrt{3}$ from Ox , and its radius vector makes an angle of 45° with Oz . Find its coördinates.

28. A point is at the distance $2\sqrt{10}$ from Oz , $2\sqrt{7}$ from Oy , and its radius vector has $l = \frac{1}{4}$. Find the point. *Ans.* $(2, \pm 6, \pm 2\sqrt{6})$.

Find the direction cosines of the line joining the given points.

29. $(3, 4, -2)$, $(1, 5, 3)$.

30. $(2, 0, 5)$, $(-3, -1, 3)$.

31. $(-\frac{2}{3}, \frac{2}{3}, 1)$, $(1, \frac{1}{3}, \frac{2}{3})$.

32. $(\frac{3}{2}, \frac{5}{4}, -\frac{1}{4})$, $(\frac{3}{4}, -\frac{1}{2}, 2)$.

33. Find the direction cosines of the sides of the triangle having the vertices $(4, 1, -2)$, $(6, 0, 4)$, $(-2, 3, -4)$.

34. Show that the line joining the points $(3, 2, -1)$, $(4, -1, 1)$ is parallel to the line joining the points $(4, 3, -3)$, $(2, 9, -7)$.

Solve the following by a new method.

35. Ex. 19, p. 207.

36. Ex. 20, p. 207.

37. Ex. 21, p. 207.

38. Ex. 22, p. 207.

39. Ex. 33, p. 208.

40. Ex. 34, p. 208.

41. Prove that the points $(5, 4, 6)$, $(8, 7, 6)$, $(0, -1, -2)$, $(1, 0, 2)$ form an isosceles trapezoid.

42. Prove that the points $(7, 1, 5)$, $(8, 2, -3)$, $(3, -3, 1)$, $(4, -2, 5)$ form a kite.

43. Prove that the points $(0, 3, 1)$, $(2, 5, -9)$, $(-4, -1, -3)$, $(1, 4, -7)$ form an arrowhead.

144. Projections. The *projection* of a point P upon any line is defined as the foot of the perpendicular from P to that line. The projection of a line segment P_1P_2 upon any line is the segment joining the projections of the endpoints P_1, P_2 upon that line.

The projection of a broken line upon any line is the sum of the projections of the segments forming the broken line.

The projection of a broken

line $P_1P_2 \cdots P_n$ upon any line is equal to the projection of the closing line P_1P_n upon that line. Thus, in Fig. 119,

$$L_1L_2 + L_2L_3 + L_3L_4 = L_1L_4.$$

It should be noted that the segments composing the broken line need not all lie in the same plane.

145. Angle between two lines. To find the angle ϕ between any two lines λ_1, λ_2 intersecting at the origin, let us denote the direction cosines of λ_1 by l_1, m_1, n_1 , those of

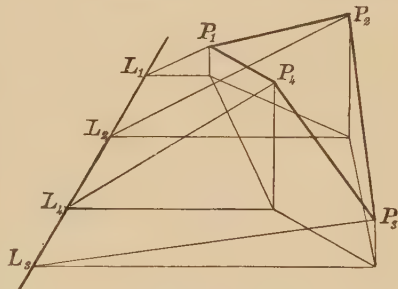


FIG. 119

λ_2 by l_2, m_2, n_2 , and choose on λ_1 any point $P : (x, y, z)$ with radius vector ρ . Then

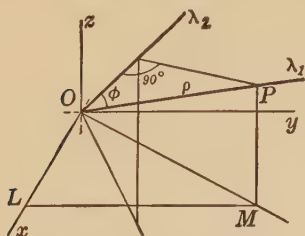


FIG. 120

$$\begin{aligned} OL &= x = l_1\rho, \\ LM &= y = m_1\rho, \\ MP &= z = n_1\rho. \end{aligned}$$

Now the projection on λ_2 of the broken line $OLMP$ equals the projection on λ_2 of the closing line OP :

$$OP \cos \phi = OL \cdot l_2 + LM \cdot m_2 + MP \cdot n_2,$$

or

$$\rho \cos \phi = l_1\rho l_2 + m_1\rho m_2 + n_1\rho n_2;$$

that is,

$$(1) \quad \cos \phi = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

If the two lines have direction components a_1, b_1, c_1 and a_2, b_2, c_2 , the direction cosines may be found by § 141. Substituting in formula (1), we find

$$(2) \quad \cos \phi = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Of course these results apply at once to any two intersecting lines. Further, the angle between two non-intersecting lines is defined as being *equal to the angle between two intersecting lines that are respectively parallel to the given lines*. With this convention the formulas give the angle between any two lines in space.

On account of the ambiguity of sign mentioned in §§ 140, 141, each of the above formulas will give either a positive or a negative result, according to the way we happen to choose the signs. This corresponds, of course, to the fact that there are two angles "between two lines," one the supplement of the other.

Example: Find the angle between the lines joining the points (3, 1, 2), (4, 0, 4) and (-2, 4, 4), (0, -1, 3).

By § 143, the direction components of the lines are respectively 1, -1, 2 and 2, -5, -1. By (2), we find

$$\cos \phi = \frac{2 + 5 - 2}{\sqrt{6} \cdot \sqrt{30}} = \frac{1}{6}\sqrt{5}.$$

146. Perpendicular lines. In formula (1), § 145, if $\cos \phi = 0$, then $\phi = 90^\circ$, and we have the

THEOREM: *Two lines having the direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 are perpendicular if*

$$(1) \quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0.$$

COROLLARY: *Two lines with direction components a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular if*

$$(2) \quad a_1 a_2 + b_1 b_2 + c_1 c_2 = 0.$$

EXERCISES

1. Find the angle between two lines whose direction components are 2, 1, -2 and 1, 1, 0. *Ans.* 45° .

2. Find the angle between two lines whose direction components are 1, 2, 3 and 2, 2, -1. *Ans.* $\arccos \frac{1}{4}\sqrt{14}$.

3. Find the angle between the radius vectors of the points (3, -4, 5) and (-1, -1, 0). *Ans.* $\arccos \frac{1}{10}$.

4. Find the angle between the radius vectors of the points (2, -2, 2) and (2, 1, 2). *Ans.* $\arccos \frac{1}{3}\sqrt{3}$.

5. Find the angle between the lines joining the points (3, 1, -2), (1, -2, 4) and (-4, 8, 0), (5, 2, -2). *Ans.* $\arccos \frac{1}{7}$.

6. Find the angle between the lines joining the points (3, 1, -2), (4, 0, -4) and (4, -3, 3), (6, -2, 2). *Ans.* $\frac{1}{3}\pi$.

Solve the following by using the corollary of § 146.

7. Ex. 15(a), p. 207.

8. Ex. 16(a), p. 207.

9. Ex. 17(a), p. 207.

10. Ex. 18(a), p. 207.

11. Ex. 36, p. 208.

12. Ex. 41, p. 213.

13. Prove in two ways that the points (1, 3, -3), (2, 2, -1), (3, 4, -2) form an equilateral triangle.

CHAPTER XVI

THE PLANE

147. The locus of an equation. If x , y , and z are connected by an equation of any form, we may assign values at pleasure to two of the variables and compute the third, thus determining certain sets of values of x , y , and z satisfying the equation. Each of these sets of values may be considered as the cartesian coördinates of a point. The points whose coördinates satisfy the equation are not scattered at random throughout space; instead, they form a definite *surface*, called the *locus* of the equation:

In space of three dimensions, the locus of an equation is a surface containing those points, and only those points, whose coördinates satisfy the equation.*

Examples: (a) If the coördinates satisfy the equation

$$x = y,$$

the point must be equidistant from the yz - and zx -planes. Hence the locus is a plane through the z -axis bisecting the angle between the yz - and zx -planes.

(b) If the coördinates of a point satisfy the equation

$$x^2 + y^2 + z^2 = 25,$$

the point must be at the distance 5 from the origin. Hence the locus is a sphere of radius 5 with center at O .

As suggested above, to determine points on a surface, we *assign values to two of the variables and compute the third*. In particular, to find the intercepts on the axes, we *equate two of the variables to 0 and solve for the third*: to find the x -intercepts, set y and z equal to 0, etc.

* Exceptionally, a surface may reduce to a single line or a single point, etc.; or there may be no locus at all. Such forms are of little importance.

148. Planes parallel to a coördinate plane. The equation

$$x = k$$

represents a plane parallel to the yz -plane at a distance k from it; for that plane contains all those points, and only those points, whose x -coördinate is k . An analogous result holds for an equation of the same form in y or in z . Hence:

An equation of the first degree in one variable represents a plane parallel to the plane of the other two variables; and conversely.

149. Plane sections of a surface; traces. A plane and a surface intersect in general in a curve, called the *section* of the surface by the plane. Of particular importance are the sections of a surface by the coördinate planes; these sections will be called for brevity the *traces* of the surface.

If in the equation of the surface we substitute $z = k$, the resulting equation in x and y , considered as the equation of a curve in the plane $z = k$, represents the section of the surface by that plane. In particular, *to obtain the xy -trace, we set $z = 0$.* Similarly for the other traces.

150. Normal form of the equation of a plane. Let $P : (x, y, z)$ be any point of the plane RST , and N the foot of the perpendicular from O upon the plane. Denote the length of the normal ON by p and its direction cosines by l, m, n . By (2), § 142, the coördinates of N are (lp, mp, np) , so that by § 143 the direction components of NP are $x - lp, y - mp, z - np$. By § 146,

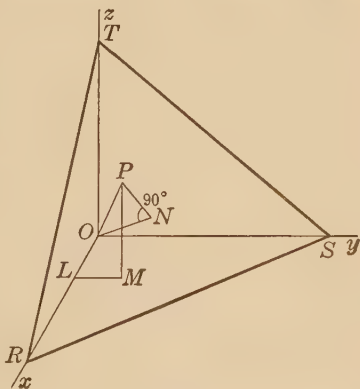


FIG. 121

the lines are perpendicular if and only if

$$l(x - lp) + m(y - mp) + n(z - np) = 0;$$

this is therefore the equation of the plane. Simplifying, we get

$$lx + my + nz = l^2p + m^2p + n^2p,$$

or

$$(1) \quad lx + my + nz = p.$$

This is called the *normal form* of the equation of the plane. For definiteness, the convention is adopted that p shall be always positive.

The above proof is easily modified to take care of the case in which the plane passes through the origin, and the same equation is obtained. Hence, since formula (1) applies to any plane, no matter how situated, we have proved the

THEOREM: *The equation of a plane is always of the first degree.*

151. General form; reduction to normal form. Every equation of the first degree can be written in the form

$$(1) \quad Ax + By + Cz + D = 0.$$

Let us transpose the constant term to the right member and make it positive (by changing all signs if necessary), and then divide through by $\sqrt{A^2 + B^2 + C^2}$:

$$(2) \quad \pm \frac{A}{\sqrt{A^2 + B^2 + C^2}} x \pm \frac{B}{\sqrt{A^2 + B^2 + C^2}} y \\ \pm \frac{C}{\sqrt{A^2 + B^2 + C^2}} z = \frac{\mp D}{\sqrt{A^2 + B^2 + C^2}}.$$

Now the coefficients of x , y , and z are the direction cosines of a certain line, since the sum of their squares is unity: it

follows that (2) is *the equation of a plane in the normal form*.

From the fact that this reduction can always be carried out (since $\sqrt{A^2 + B^2 + C^2}$ cannot be 0) we deduce the

THEOREM: *Every equation of the first degree represents a plane.*

Further, we have the

RULE: *To reduce the equation of any plane*

$$Ax + By + Cz + D = 0$$

to the normal form, divide by $\sqrt{A^2 + B^2 + C^2}$ and choose signs so that the constant is positive in the right member.

152. Perpendicular line and plane. From the fact that the coefficients A, B, C in the general equation of the plane are proportional to the direction cosines of the normal, we obtain at once the very important

THEOREM: *If a line and plane are perpendicular, the coefficients of x, y, z in the equation of the plane may be taken as direction components of the line, and vice versa.*

A typical application is furnished by the following

Example: Write the equation of a plane perpendicular to the radius vector of the point (3, 1, 2) and passing through (0, 4, 5).

By § 142, the direction components of the radius vector are 3, 1, 2; by the above theorem, these numbers may be taken as the coefficients in the equation of the plane. The result is

$$3x + y + 2z = 14,$$

the right member being necessarily the value assumed by the left member when the coördinates (0, 4, 5) are substituted.

EXERCISES

For the given plane, (a) find the intercepts on the axes and at least one other point; (b) write the equations of the traces; (c) draw the traces.

- | | |
|--------------------------|--------------------------|
| ✓ 1. $x + 3y + 4z = 6$. | 2. $3x + 2y + 5z = 15$. |
| 3. $x + y - 2z = 4$. | 4. $2x + y - z = 2$. |
| 5. $2x - y + z = 8$. | 6. $4x - 2y + 3z = 12$. |
| ✓ 7. $x + 2y = 3$. | 8. $2y + 3z = 1$. |

Reduce the given equation to the normal form; determine the direction cosines of the normal and the distance from the origin.

- | | |
|-----------------------------|-----------------------------|
| 9. $3x - 2y + 6z = 21$. | 10. $4x + 8y - z = 45$. |
| 11. $x - 3y - 2z + 7 = 0$. | 12. $3x - 5y - z + 6 = 0$. |
| 13. $x - 3y + 4z = 0$. | 14. $2x + y + z = 0$. |
| 15. $3x + y + 2 = 0$. | 16. $x - 2z + 1 = 0$. |

Find the equations of the following planes.

- ✓ 17. At a distance 3 from O , with the normal having direction components 2, -1, -2. *Ans.* $2x - y - 2z = \pm 9$.

18. At a distance $2\sqrt{6}$ from O , with the normal having direction components -1, 1, 2. *Ans.* $x - y - 2z = \pm 12$.

19. Through (3, 1, -1) perpendicular to the radius vector of that point. *Ans.* $3x + y - z = 11$.

- ✓ 20. Through (-2, 1, -3) perpendicular to the radius vector of that point. *Ans.* $2x - y + 3z + 14 = 0$.

21. With the point (-2, 4, 2) as the foot of the normal from O .

22. With the point (3, -6, 2) as the foot of the normal from O .

23. Through (2, 5, -2) perpendicular to the radius vector of the point (1, -3, -4). *Ans.* $x - 3y - 4z + 5 = 0$.

24. Through (-3, 1, 4) perpendicular to the radius vector of the point (1, -2, 2). *Ans.* $x - 2y + 2z = 3$.

25. Through the origin perpendicular to the line joining (3, -1, 4), (-2, 2, 1). *Ans.* $5x - 3y + 3z = 0$.

26. Through (0, 1, 0) perpendicular to the line joining (2, -3, 1), (3, 4, 0). *Ans.* $x + 7y - z = 7$.

27. Perpendicular to the radius vector of (1, 3, -2) and passing at a distance 2 from the origin. *Ans.* $x + 3y - 2z = \pm 2\sqrt{14}$.

28. Perpendicular to the radius vector of (3, -2, 2) and passing at a distance 3 from the origin. *Ans.* $3x - 2y + 2z = \pm 3\sqrt{17}$.

29. The base of an isosceles triangle joins the points (3, 1, -2), (-1, 3, 0). Find the locus of the third vertex by § 152.

30. The base of an isosceles triangle connects the points $(4, 2, 0)$, $(2, -6, 2)$. Find the locus of the third vertex by § 152.

31. Solve Ex. 29 by another method.

32. Solve Ex. 30 by another method.

33. One side of a right triangle joins the points $(1, 5, -1)$, $(-2, 3, 0)$, with the right angle at the latter point. Find the locus of the third vertex.

34. One side of a right triangle joins the points $(0, 5, -3)$, $(2, 2, 1)$, with the right angle at the latter point. Find the locus of the third vertex.

35. Solve Ex. 33 by another method.

36. Solve Ex. 34 by another method.

37. Derive the normal form from the fact that, in Fig. 121, the projection of the broken line $OLMPN$ upon the normal must equal ON .

38. Derive the normal form from the fact that the midpoint of OP is equidistant from O and N . Does the derivation hold in all cases?

39. Derive the normal form by applying the Theorem of Pythagoras to the triangle ONP . Does the derivation hold in all cases?

40. Find the coördinates of the point N' such that N (Fig. 121) is the midpoint of ON' .

41. Using Ex. 40, derive the normal form by finding the locus of points equidistant from O and N' . Does the derivation hold in all cases?

153. Parallel planes. If the equations of two planes differ only in the constant term, the same must be true after reduction to the normal form. Hence the normals to the two planes have the same direction cosines and are parallel, from which it follows that *the planes are parallel*.

Conversely, *if two planes are parallel, their equations can be made to differ only in the constant term.*

Example: Write the equation of a plane through $(1, 2, 1)$ parallel to the plane

$$2x + 2y + z = 3.$$

The answer is

$$2x + 2y + z = 7,$$

the right member being written down immediately from the fact that it must be the value assumed by the left member when the coördinates $(1, 2, 1)$ are substituted.

EXERCISES

Find the equations of the following planes.

1. Through $(-2, 3, 4)$ parallel to the plane $2x - y - 2z = 5$.
 2. Through $(3, 4, -2)$ parallel to the plane $x - y + 3z = 4$.
 3. Through $(5, 0, 2)$ parallel to the plane $z = 2x + 3y$.
 4. Through $(1, 5, -2)$ parallel to the plane $y = 3x + 4z + 3$.
 5. Parallel to the plane $6x + 3y - 2z = 14$ and (a) half as far from the origin; (b) at the distance 3 from the origin; (c) at the same distance from the origin.
Ans. (a) $6x + 3y - 2z = \pm 7$.
 6. Parallel to the plane $3x - y + z = 6$ and (a) twice as far from the origin; (b) at the same distance from the origin; (c) at the distance 2 from the origin.
Ans. (c) $3x - y + z = \pm 2\sqrt{11}$.
 7. Parallel to the plane $x - 2y + 2z + 6 = 0$ and (a) 2 units farther from the origin; (b) 1 unit nearer the origin; (c) at a distance 5 from the given plane.
Ans. (a) $x - 2y + 2z = \pm 12$.
 8. Parallel to the plane $x - 4y - 8z + 27 = 0$ and (a) 3 units farther from the origin; (b) at the distance 1 from the given plane; (c) 2 units nearer the origin.
Ans. (c) $x - 4y - 8z = \pm 9$.
 9. Parallel to the plane $x - y - z + 5 = 0$ and passing at the distance $2\sqrt{3}$ from $(1, 2, -2)$.
Ans. $x - y - z = 1 \pm 6$.
 10. Parallel to the plane $2x - 3y - 5z + 1 = 0$ and passing at the distance 3 from $(-1, 3, 1)$.
 11. Parallel to the plane $2x + 2y + z = 0$ and passing (a) half as far from $(1, 3, -2)$; (b) 2 units farther from $(1, 3, -2)$; (c) at a distance 3 from $(1, 3, -2)$.
Ans. (b) $2x + 2y + z = 6 \pm 12$.
 12. Parallel to the plane $2x - 6y + 9z = 0$ and (a) twice as far from $(5, 1, 2)$; (b) 1 unit farther from $(5, 1, 2)$; (c) at a distance 3 from $(5, 1, 2)$.
- Find the distance between the given planes.
13. $2x - y + 3z + 5 = 0$, $2x - y + 3z + 2 = 0$.
 14. $3x - y - z = 6$, $6x - 2y - 2z = 5$.
 15. $x + 3y + 3z = 4$, $3x + 9y + 9z + 8 = 0$.
 16. $2x - 5y + z + 1 = 0$, $2x - 5y + z = 3$.
 17. A sphere of radius $\sqrt{5}$ rolls on the plane $4x - 2y + 5z = 2$. Find the locus of its center.
 18. A sphere of radius $2\sqrt{21}$ rolls on the plane $4x + y - 2z = 3$. Find the locus of its center.
 19. Two faces of a cube lie in the planes $x + 2y + 2z - 3 = 0$, $3x + 6y + 6z + 2 = 0$. Find the volume of the cube.

20. Show that the planes $x + y - z + 1 = 0$, $x + y - z - 1 = 0$, $3x - 2y + z + 3 = 0$, $3x - 2y + z + 2 = 0$, $x + 4y + 5z + 20 = 0$, $x + 4y + 5z = 1$ form a box, and find its volume. *Ans.* $V = 1$.

154. Plane through a given point. If the plane

$$Ax + By + Cz + D = 0$$

is to pass through the point (x_1, y_1, z_1) , those coördinates must satisfy the equation, and we have

$$Ax_1 + By_1 + Cz_1 + D = 0.$$

By subtraction, we find that *the equation of any plane through the point (x_1, y_1, z_1) may be written in the form*

$$(1) \quad A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.$$

155. Plane determined by three points. From the fact that the general equation of the plane contains three essential constants—viz. the ratios of any three of the quantities A, B, C, D to the fourth—we conclude that *a plane is determined by three points*, or by any set of three conditions which when expressed analytically give three equations to determine the essential ratios. (Cf. § 38.)

The direct method of finding the equation of a plane determined by three points is to substitute the coördinates of the points in turn in the equation

$$Ax + By + Cz + D = 0,$$

thus obtaining three equations to solve for three of the constants in terms of the fourth. The solution may, however, be expedited by use of (1), § 154, as in the following

Example: Find the equation of the plane through the points $(2, 4, 3)$, $(1, 3, 1)$, $(-1, -1, -4)$.

The equation of any plane through $(2, 4, 3)$ is

$$(1) \quad A(x - 2) + B(y - 4) + C(z - 3) = 0.$$

Substitute the coördinates of the other points in (1):

$$\begin{aligned} -A - B - 2C &= 0, \\ -3A - 5B - 7C &= 0. \end{aligned}$$

Solving for A and C in terms of B , we find

$$A = 3B, \quad C = -2B.$$

By substitution in (1) the equation of the plane is found to be

$$3B(x - 2) + B(y - 4) - 2B(z - 3) = 0,$$

or

$$3x + y - 2z = 4.$$

156. Angle between two planes. The angle between two planes is evidently equal to the angle between their normals. Hence the angle between two planes is found by substituting the direction components of the normals in formula (2), § 145.

Example: Find the angle between the planes

$$x + 2y - 3z = 8, \quad 2x - 4y - 5z = 6.$$

By § 152, the two sets of direction components are 1, 2, -3 and 2, -4, -5. Hence *

$$\cos \phi = \frac{2 - 8 + 15}{\sqrt{14} \cdot \sqrt{45}} = \frac{9}{3\sqrt{70}} = \frac{3}{\sqrt{70}}.$$

157. Perpendicular planes. Two planes

$$\begin{aligned} A_1x + B_1y + C_1z + D_1 &= 0, \\ A_2x + B_2y + C_2z + D_2 &= 0 \end{aligned}$$

are perpendicular if their normals are perpendicular to each other. The direction cosines of the normals are proportional to A_1, B_1, C_1 and A_2, B_2, C_2 respectively. Applying the condition of perpendicularity (Corollary, § 146) to these two lines, we obtain the following

* The sign is ambiguous. See § 145.

THEOREM: *Two planes*

$$A_1x + B_1y + C_1z + D_1 = 0,$$

$$A_2x + B_2y + C_2z + D_2 = 0$$

are perpendicular if

$$(1) \quad A_1A_2 + B_1B_2 + C_1C_2 = 0;$$

and conversely.

Example: Find the equation of a plane through the points (1, 1, 2), (2, 4, 3), and perpendicular to the plane

$$(2) \quad x - 3y + 7z + 5 = 0.$$

The equation of any plane through (1, 1, 2) is

$$(3) \quad A(x - 1) + B(y - 1) + C(z - 2) = 0.$$

Substituting the coördinates (2, 4, 3) in (3), we get

$$(4) \quad A + 3B + C = 0.$$

The condition of perpendicularity (1) applied to the planes (2) and (3) gives

$$(5) \quad A - 3B + 7C = 0.$$

From (4) and (5) we find

$$A = -4C, \quad B = C,$$

whence the equation of the plane is

$$-4C(x - 1) + C(y - 1) + C(z - 2) = 0,$$

or

$$4x - y - z = 1.$$

158. Planes perpendicular to a coördinate plane. An equation of the first degree in which z is missing, say

$$Ax + By + D = 0,$$

represents a plane perpendicular to the xy -plane. For the equation of the xy -plane is

$$z = 0,$$

and condition (1), § 157, applied to these two planes gives

$$A \cdot 0 + B \cdot 0 + 0 \cdot 1 = 0,$$

which is true for all values of A and B . A similar result holds for equations in which x or y is missing. Hence the

THEOREM: *An equation of the first degree in two variables represents a plane perpendicular to the plane of those two variables; and conversely.*

A different line of argument by which this theorem may be proved will appear in § 183, and might equally well have been used at this point.

EXERCISES

Find the equation of the plane through the given points.

1. $(2, 3, -3), (1, 1, -2), (-1, 1, 4)$. *Ans.* $3x - y + z = 0$.

2. $(2, 2, 2), (3, 1, 1), (6, -4, -6)$. *Ans.* $x + 2y - z = 4$.

3. $(3, 3, 1), (-3, 2, -1), (8, 6, 3)$. *Ans.* $4x + 2y - 13z = 5$.

4. $(-1, 1, -5), (3, -1, 5), (-2, 3, 0)$. *Ans.* $5x + 5y - z = 5$.

5. $(0, 1, -1), (2, 4, -3), (3, -5, -6)$.
Ans. $27x - 4y + 21z + 25 = 0$.

6. $(1, 1, 2), (3, 2, 1), (5, 3, 0)$. Explain the meaning of the result.

Find the angle between the given planes.

7. $x + 2y + z = 3, x + y + 4 = 0$. *Ans.* $\frac{1}{6}\pi$.

8. $x + 2y + 3z = 2, 3x - y + 2z = 0$. *Ans.* $\frac{1}{3}\pi$.

9. $x + 2y - z = 6, x - 2y - 2z = 2$. *Ans.* $\arccos \frac{1}{18}\sqrt{6}$.

10. $3x - z + 2 = 0, x + 3y = 5$. *Ans.* $\arccos \frac{3}{10}$.

11. $3x + 5y + 4z = 2$ and the zx -plane. *Ans.* $\frac{1}{4}\pi$.

12. $x + 3y + 4z = 6$ and the xy -plane. *Ans.* $\arccos \frac{2}{13}\sqrt{26}$.

Find the equations of the following planes.

13. Through the points $(1, 2, 3), (3, 2, -1)$, perpendicular to the plane $3x + 2y + 6z + 4 = 0$. *Ans.* $2x - 6y + z + 7 = 0$.

14. Through the points $(0, 2, 3), (5, -1, 4)$, perpendicular to the plane $x + 2y + 3 = 0$. *Ans.* $2x - y - 13z + 41 = 0$.

15. Through the points $(2, 1, 1), (3, 2, 2)$, perpendicular to the plane $x + 2y - 5z = 3$. *Ans.* $7x - 6y - z = 7$.

16. Through $(4, 1, 2)$, $(1, 3, -1)$, perpendicular to the yz -plane.

17. Through $(4, 1, 0)$, perpendicular to the planes $2x - y - 4z = 6$,
 $x + y + 2z = 3$. *Ans.* $2x - 8y + 3z = 0$.

18. Through $(1, 1, 2)$, perpendicular to the planes $2x - 2y - 4z = 3$,
 $3x + y + 6z = 4$. *Ans.* $x + 3y - z = 2$.

19. Perpendicular to the planes $y = 3x + z$, $x + 5y + 3z = 0$, and
 passing at a distance $\sqrt{6}$ from the origin. *Ans.* $x + y - 2z = \pm 6$.

20. Perpendicular to the planes $z = 4y - x$, $3x + 4y + z = 2$, and
 passing at a distance 1 from the origin. *Ans.* $4x - y - 8z = \pm 9$.

21. Perpendicular to the planes $2x - y + z = 1$, $3x + y = 6z$, and
 passing at a distance 2 from $(1, 2, -1)$. *Ans.* $x + 3y + z = 6 \pm 2\sqrt{11}$.

22. Perpendicular to the planes $x = 2y - 2z$, $x - 3y + z = 1$, and
 passing at a distance $\sqrt{2}$ from $(0, 4, 2)$. *Ans.* $4x + y - z = 2 \pm 6$.

23. Through $(1, -2, -1)$, perpendicular to the plane $x - 3y = 4z$,
 and passing at a distance 1 from O .

Ans. $6x - 2y + 3z = 7$, $4y - 3z + 5 = 0$.

24. Prove that the equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

represents the plane determined by the points (x_1, y_1, z_1) , (x_2, y_2, z_2) ,
 (x_3, y_3, z_3) .

25. In Ex. 24, what happens if the minors of all the elements in the
 first row are 0? Explain geometrically.

26. Use the formula of Ex. 24 to solve Exs. 1-6.

CHAPTER XVII

THE STRAIGHT LINE

159. Representation of a curve in space. Two surfaces intersect in general in a curve. If the equations of the two surfaces be considered as *simultaneous*, their locus consists of all points lying on both surfaces. Hence:

The locus of two simultaneous equations is a curve, viz. the curve of intersection of the surfaces represented by the two equations separately.

160. The straight line. Two equations of first degree represent, of course, two planes. When considered as *simultaneous*, the equations represent all points common to the two planes. From the fact that two planes (if not parallel) intersect in a straight line, we deduce the following

THEOREM: *The locus of two simultaneous equations of the first degree is a straight line.*

Thus the general form of the equations of a straight line is

$$(1) \quad \begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0. \end{cases}$$

Since through any line an infinite number of planes may be passed, it is obvious that a straight line may be represented by two simultaneous equations in an infinite number of ways—viz. by the equations of any two planes through the line. Of these various equations there are usually one or two pairs especially simple and convenient (see §§ 162, 163).

To locate a point on a straight line, we *assign a value to one of the variables and solve the resulting simultaneous equations for the values of the other two*. In particular, the *piercing points* in the coördinate planes are found by *setting x , y , and z in turn equal to 0* and solving the resulting equations for the other two variables.

161. Planes through a given line. Let the equations of a straight line be denoted by

$$(1) \quad \begin{cases} u = 0, \\ v = 0, \end{cases}$$

where u and v represent any expressions of the first degree in x , y , and z . Then the equation

$$(2) \quad u + kv = 0,$$

where k is a constant, is an equation of the first degree and thus represents a plane; further, since (2) is satisfied whenever both the equations (1) are satisfied, the plane (2) contains all points of the line (1). (Cf. § 41.) Hence the

THEOREM: *If*

$$u = 0, \quad v = 0$$

are the equations of any line, the equation

$$(3) \quad u + kv = 0,$$

where k is a constant, represents a plane through the given line.

It is geometrically obvious that a plane containing a given line may be made to pass through a given point, or to be perpendicular to another plane, or to pass at a given distance from a fixed point, etc. The truth of these statements appears analytically from the fact that equation (3) contains one undetermined constant.

Any two equations of the family (3) may of course be used to represent the line, instead of the original pair.

Example: Find the equation of the plane through the line

$$2x - 2y + 3z = 1, \quad x - 5y - z = 2$$

and perpendicular to the plane

$$(4) \quad 2x + y + 4z = 5.$$

The required equation must be of the form

$$2x - 2y + 3z - 1 + k(x - 5y - z - 2) = 0,$$

or

$$(5) \quad (2 + k)x - (2 + 5k)y + (3 - k)z - 1 - 2k = 0.$$

By § 157, planes (4) and (5) will be perpendicular if

$$2(2 + k) - 1(2 + 5k) + 4(3 - k) = 0.$$

This gives

$$k = 2;$$

substituting in (5), we have the answer

$$4x - 12y + z - 5 = 0.$$

162. Projecting planes. The planes through a line perpendicular to the coördinate planes are called the *projecting planes*, and their traces are the *projections*, of the line on the coördinate planes. It is often convenient, particularly in making a sketch, to represent a line by the equations of two of its projecting planes.

If in (3), § 161, we choose k so that the term in x drops out, which amounts merely to eliminating x between the equations of the line, the resulting equation represents a plane through the given line perpendicular to the yz -plane (§ 158). In this way we establish the

RULE: *To obtain the yz -projecting plane of a line, eliminate x between the equations of the line; similarly for the other projecting planes.*

Example: Draw the line

$$4x + 4y + 3z = 14,$$

$$x - 2y + 3z = 2.$$

Eliminating z and y in turn, we find the xy - and zx - projecting planes:

$$x + 2y = 4, \quad 2x + 3z = 6.$$

The xy -traces of these planes are the lines LQ , MP_1 ; the yz -traces are LP_2 , NP_2 . The points P_1 , P_2 are evidently the xy - and yz -piercing points of the given line.

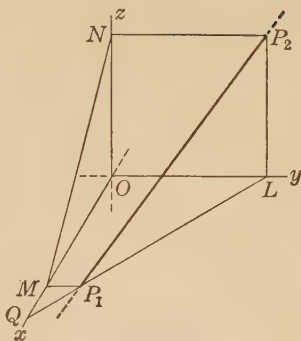


FIG. 122

EXERCISES

Find the piercing points of the following lines (§ 160).

1. $5x + 2y + 3z = 9$, $4x + 3y + z = 10$.

Ans. $(0, 3, 1)$, $(3, 0, -2)$, $(1, 2, 0)$

2. $5x - 3y + 9z + 3 = 0$, $7x - 9y + 3z = 15$.

Ans. $(0, -2, -1)$, $(3, 0, -2)$, $(-3, -4, 0)$.

3. $x + 2y + 4z = 6$, $2x - y - z = 2$.

4. $x + 5y - z + 3 = 0$, $3x + 2y + z = 1$.

5. $3x + y - z + 2 = 0$, $x + 2y + 2z = 4$.

6. $x + 3y - 2z = 3$, $2x - 2y + 3z + 2 = 0$.

7. $3x + y + 2z = 0$, $x + 4y = 0$.

8. $x - y + 3z = 0$, $3x + 2y - z = 0$.

Find the equations of the following planes.

9. Through the line $x + y = 2$, $z = 4$ and the point $(2, 4, 0)$. Draw the figure.

Ans. $x + y + z = 6$.

10. Through the line $x + 2z = 2$, $y = 2$ and the point $(4, 1, 0)$. Draw the figure.

Ans. $x + 2y + 2z = 6$.

11. Through the line $3x - y + 2z = 6$, $2x + y - z = 5$ and the point $(1, 2, -2)$.

12. Through the line $x + 4y - z + 2 = 0$, $3x - 5y + 2z + 2 = 0$ and the point $(-3, 2, -1)$.

13. Through the line $3x + y - z = 3$, $x + 2y + 4z + 4 = 0$ and perpendicular to the plane $x - 2y - z = 5$. *Ans.* $23x + 11y + z = 13$.

14. Through the line $x - y = 3$, $y - 2z = 4$ and perpendicular to the plane $3x - 5y + 6z = 2$. *Ans.* $17x - 9y - 16z = 83$.

15. Through the line $x - y + z = 0$, $2x + y + z = 2$ and perpendicular to the plane $x + 2y + 2z = 4$.

16. Through the line $3x + y - z = 4$, $x + 2y + z = 1$ and perpendicular to the plane $2x - y - z = 5$.

17. Through the line of Ex. 15, perpendicular to the zx -plane.

18. Through the line of Ex. 16, perpendicular to the xy -plane.

19. Through the line $y - 2z = 6$, $x + y = 0$ and passing at a distance 2 from the origin. *Ans.* $x + 2y - 2z = 6$, $2x + y + 2z + 6 = 0$.

20. Through the line $x - 5z = 6$, $y = z$ and passing at a distance $\sqrt{2}$ from the origin. *Ans.* $x - 4y - z = 6$, $x - y - 4z = 6$.

21. Through the line $y - 5z + 1 = 0$, $x + y = 2$ and passing at a distance $\frac{1}{2}\sqrt{2}$ from O . *Ans.* $3x + 4y - 5z = 5$, $4x + y + 15z = 11$.

22. Through the line $2x - z = 5$, $x + y = 2$ and passing at a distance $\sqrt{3}$ from the origin. *Ans.* $x - y - z = 3$, $7x + 5y - z = 15$.

23. Show that the line $2x + y - 3z = 4$, $3x - y + 2z = 2$ lies in the plane $4x + 7y - 19z = 16$.

24. Show that the line $2x + 3y - z = 4$, $3x - y - 2z = 1$ lies in the plane $x - 15y - 2z + 13 = 0$.

Write the equations of the following lines.

25. Through $(2, -3, -2)$ parallel to the line $3x - 2y + 3z + 3 = 0$, $x + y - 2z = 4$. *Ans.* $3x - 2y + 3z = 6$, $x + y - 2z = 3$.

26. Through $(-3, 1, -2)$ parallel to the line $x + 2y - 3z = 4$, $x - y - 4z = 0$. *Ans.* $x + 2y - 3z = 5$, $x - y - 4z = 4$.

27. Through $(0, -5, -1)$ parallel to the line $12x - y + 3z + 2 = 0$, $3x + y + z = 4$.

28. Through $(4, -1, -3)$ parallel to the line $3x + 4y = 2z$, $x = 3y$.

Find the projecting planes of the following lines, and draw the lines.

29. $x + y + 3z = 3$, $2x + y + 4z = 4$.

30. $4x + y + z = 4$, $3x + 3y + 2z = 6$.

31. $4x + 3y + z = 18$, $2x - 3y + 2z = 0$.

32. $4x + y + 3z = 9$, $5x + y + 6z = 15$.

33. $2x + y - z = 3$, $3x + 2y - z = 6$.

34. $3x - 2y - z = 0$, $x - y + 2z = 0$.

35. $2x + 4y + z = 0$, $x + y + z = 0$.

36. $x + 3y + 4z = 15$, $x - 6y - 8z + 21 = 0$.

37. $2x + y + z = 3$, $6x + 3y - 2z + 1 = 0$.

163. Symmetric form. The equations of the line through (x_1, y_1, z_1) having direction components a, b, c may be found as follows. If (x, y, z) is any point of the line, the direction cosines are also proportional to $x - x_1, y - y_1, z - z_1$. Since two sets of numbers proportional to the same third set are proportional to each other, we have*

$$(1) \quad \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

Formula (1) is called the *symmetric form*; for most purposes it is more convenient than any other.

Example: Write the equations of the line joining the points $(2, 3, 1), (1, -3, -1)$.

By § 143, the direction components are 1, 6, 2; hence the equations of the line are

$$(2) \quad \frac{x - 2}{1} = \frac{y - 3}{6} = \frac{z - 1}{2}.$$

Given the equations of a line in the symmetric form, it is frequently necessary to obtain two equations in the ordinary or general form (§ 160). As an example, we will do this for the line (2). Disregarding the third member and clearing of fractions, we get

$$(3) \quad 6x - y = 9.$$

Disregarding the second member, we find

$$(4) \quad 2x - z = 3.$$

Equations (3) and (4) represent the line; if more equations are needed, combinations of these may be made (§ 161).

* It may be objected that in (1) there are three equations, viz.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}, \quad \frac{x - x_1}{a} = \frac{z - z_1}{c}, \quad \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

These equations, however, are not independent, since any one follows from the other two. These equations of course represent three planes through the line; they are in fact the projecting planes.

164. Determination of direction cosines; reduction to the symmetric form. Being given the equations of a line in the general form, it is often necessary to reduce them to the symmetric form, or at least to find a set of direction components. This may be done, in general, as in the

Example: Reduce the equations

$$\begin{aligned} 4x + 4y + 3z &= 14, \\ x - 2y + 3z &= 2 \end{aligned}$$

to the symmetric form (cf. the example of § 162).

Find any two of the projecting planes: for instance,

$$x + 2y = 4, \quad 2x + 3z = 6.$$

These equations contain one variable in common; *solve for that variable and equate values:*

$$x = -2y + 4 = \frac{-3z + 6}{2}.$$

Divide through by such a number as to reduce the coefficient of each variable to unity—in this case by -6 :

$$\frac{x}{-6} = \frac{y - 2}{3} = \frac{z - 2}{4}.$$

165. Line parallel to a coördinate plane. If the line is parallel to a coördinate plane, the above process fails, since the equations of the projecting planes will not contain one variable in common. In that case, we may find the piercing points in the other two planes, and then employ § 143.

Example: Reduce the equations

$$\begin{aligned} 2x + 2y + 3z &= 8, \\ x + 4y + 6z &= 10 \end{aligned}$$

to the symmetric form.

The projecting planes, found by the usual method, are

$$(1) \quad x = 2,$$

$$(2) \quad 2y + 3z = 4.$$

Equation (1) shows that the line is parallel to the yz -plane. The piercing points in the other coördinate planes are $(2, 2, 0)$, $(2, 0, \frac{4}{3})$. By § 143, the line through these points is *

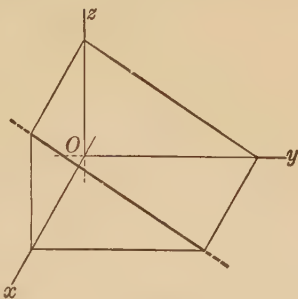


FIG. 123

$$\frac{x - 2}{0} = \frac{y - 2}{2} = \frac{z}{-\frac{4}{3}},$$

or

$$(3) \quad \frac{x - 2}{0} = \frac{y - 2}{3} = \frac{z}{-2}.$$

166. Parallel, intersecting, and skew lines. It is obvious that two lines in space do not in general lie in the same plane, in which case they are said to be *skew*. However, they may be parallel, or they may intersect.

Being given the equations of any line in the symmetric form, we may write down at once the equations of the parallel line through any point; for the denominators a , b , c in the equations of the required line must be proportional to, and may be taken equal to, the corresponding denominators in the given equations.

Being given the equations of two pairs of planes, representing two lines, we may find the point of intersection of any three of the planes: the given lines do or do not intersect according as this point does or does not lie in the fourth plane.

* Division by 0 is always excluded. Equations (3) are to be considered merely as a handy conventional way of writing (1) and (2).

EXERCISES

Write the equations of the line through the given points.

1. $(4, 2, 1), (-1, 3, 6)$.

2. $(3, -1, 4), (4, -2, -3)$.

3. $(8, -5, 6), (0, 1, 0)$.

4. $(0, -5, -2), (1, 3, 3)$.

5. $(3, -2, 4), (2, -2, -1)$.

6. $(2, 3, 1), (-2, 4, 1)$.

7. $(3, 2, 0), (3, -3, 0)$.

8. $(2, 5, 2), (-1, 5, 2)$.

9. Find the angle between the lines

$$\frac{x}{2} = \frac{y}{-1} = \frac{z-1}{0}, \quad \frac{x+2}{-1} = \frac{y-3}{1} = \frac{z-2}{2} \quad \text{Ans. } \arccos \frac{3}{\sqrt{30}}$$

10. Find the angle between the lines

$$\frac{x}{1} = \frac{y-3}{3} = \frac{z+1}{2}, \quad \frac{x-1}{3} = \frac{y-1}{-1} = \frac{z}{2} \quad \text{Ans. } \arccos \frac{7}{10}$$

11. Show that the lines

$$\frac{x-3}{3} = \frac{y+2}{4} = \frac{z-1}{2}, \quad \frac{x-1}{2} = \frac{y+3}{-2} = \frac{z+2}{1}$$

are perpendicular.

12. Show that the lines

$$\frac{x-3}{5} = \frac{y+1}{2} = \frac{z-3}{1}, \quad \frac{x-2}{1} = \frac{y-2}{4} = \frac{z+1}{-3}$$

are perpendicular.

13. Show that the lines

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{a-b}, \quad \frac{x-x_2}{-a} = \frac{y-y_2}{b} = \frac{z-z_2}{a+b}$$

are perpendicular.

14. Show that the lines

$$\frac{x-x_1}{a} = \frac{y-y_1}{a} = \frac{z-z_1}{b+c}, \quad \frac{x-x_2}{b} = \frac{y-y_2}{c} = \frac{z-z_2}{-a}$$

are perpendicular.

Reduce the following equations to the symmetric form.

15. $x + 2y + 4z + 1 = 0, x - 5y - 5z = 2$.

16. $x + y - z = 0, 2x - 2y - z = 2$.

17. $2x + 4y + z = 14, x + 2y + 3z = 2$. Ans. $\frac{x-8}{-2} = \frac{y}{1} = \frac{z+2}{0}$.

18. $6x - y + 2z = 4, 3x + 2y + z = 7$.

19. Find the angle between the lines $2z = x + y, 2x + 2y - z = 2$
and $3x + y - z = 1, 5x + 2y - z = 3$. Ans. 30° .

20. Find the angle between the lines $x + y + 3z = 1, 2x + 3y + 6z = 3$ and $x - z = 0, x + 3y - z = 5$. Ans. $\arccos \frac{1}{3}\sqrt{5}$.

21. Write the equations of the line through the point $(2, -3, 1)$ parallel to the line $\frac{x}{2} = \frac{y-1}{4} = \frac{z+3}{-1}$.

22. Write the equations of the line through the point $(0, -5, 2)$ parallel to the line $\frac{x-3}{-1} = \frac{y-2}{0} = \frac{z+2}{2}$.

23. Show that the lines $x + y - z + 1 = 0$, $x - 3y + z = 1$ and $3x - y - z = 0$, $4x - 6y + z = 2$ are parallel.

24. Show that the lines $7x + 5y - 7z = 3$, $9x + 3y - z = 0$ and $2x + y - z = 1$, $x - y + 3z + 3 = 0$ are parallel.

25. Find the equation of the plane determined by the parallel lines of Ex. 23. Ans. $x - 5y + 2z = 2$.

26. Find the equation of the plane determined by the parallel lines of Ex. 24. Ans. $29x + 7y + 3z + 3 = 0$.

27. Find the point of intersection of the intersecting lines $y = 2x$, $2x + y + 2z = 2$ and $2x - y + 2z + 2 = 0$, $3x + y + z = 4$.

28. Find the point of intersection of the intersecting lines $x + y = z$, $2x - y - 3z = 1$ and $x + 2y + z = 3$, $3x + 3y - 2z = 2$.

29. Find the equation of the plane determined by the lines of Ex. 27. Ans. $18x + y + 10z = 10$.

30. Find the equation of the plane through the lines of Ex. 28. Ans. $12x + 9y - 13z = 1$.

31. Find the distance from the point $(3, 0, 2)$ to the line $x = y$, $2x - y - z = 1$. (Pass a plane through the point and the line, then a plane through the line perpendicular to the plane just determined.) Ans. $\sqrt{6}$.

32. Find the distance from the point $(2, 3, 1)$ to the line $y + z = 1$, $2x - 3y - 2z + 4 = 0$. (Note the suggestion in Ex. 31.) Ans. 3.

167. Perpendicular line and plane. As a corollary to the theorem of § 152 we have at once the following

THEOREM: *The plane*

$$Ax + By + Cz + D = 0$$

and the line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

are perpendicular if and only if the quantities A, B, C and a, b, c are proportional.

Example: The equation of the plane through the point $(3, 2, -1)$ perpendicular to the line

$$\frac{x-2}{2} = \frac{y-1}{1} = \frac{z+3}{4}$$

is

$$2x + y + 4z = 4.$$

168. Angle between a line and a plane. The angle ϕ between a line and a plane is evidently the complement of the angle ϕ' between the line and a normal to the plane. If l_1, m_1, n_1 are the direction cosines of the line and l_2, m_2, n_2 those of the normal to the plane, we have by § 145

$$\cos \phi' = l_1 l_2 + m_1 m_2 + n_1 n_2;$$

hence the angle between the line and the plane is given by the formula

$$(1) \quad \sin \phi = l_1 l_2 + m_1 m_2 + n_1 n_2,$$

with the usual ambiguity of sign (§ 145).

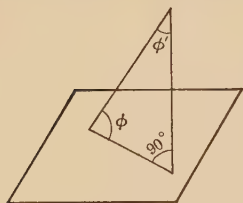


FIG. 124

169. Parallel line and plane. If a line is parallel to a plane, it is perpendicular to the normal to the plane. The direction cosines of the normal are proportional to the coefficients A, B, C in the equation of the plane. From the condition for perpendicular lines (§ 146), we deduce the

THEOREM: *The plane*

$$Ax + By + Cz + D = 0$$

and the line

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

are parallel if

$$(1) \quad aA + bB + cC = 0;$$

and conversely.

EXERCISES

Write the equations of the following lines.

1. Through $(2, -3, 3)$ perpendicular to the plane $3x + y - z = 3$.
2. Through $(3, 0, -1)$ perpendicular to the plane $x + y - 4z = 1$.
3. Through $(1, 4, 0)$ perpendicular to the plane $3y + 2z = 6$.
4. Through $(0, 2, -5)$ perpendicular to the plane $x - 5y = 2$.
5. Find the foot of the perpendicular from the origin upon the plane $2x + y + z = 12$. Draw the figure. *Ans.* $(4, 2, 2)$.
6. Find the foot of the perpendicular from the point $(3, 4, 4)$ upon the plane $x + 2y + 3z = 9$. Draw the figure. *Ans.* $(2, 2, 1)$.

Find the equations of the following planes.

7. Through $(3, 1, 1)$ perpendicular to the line $\frac{x-2}{3} = \frac{y-1}{4} = \frac{z}{6}$.

8. Through $(4, 0, -1)$ perpendicular to the line

$$\frac{x-3}{1} = \frac{y-2}{2} = \frac{z+3}{-1}.$$

9. Through $(1, 3, -2)$ perpendicular to the line $x = 2y = 3z + 3$.
Solve by two methods (§§ 157, 167).

10. Through $(4, 1, 0)$ perpendicular to the line $x - 2 = 2y - 3 = z$.

11. Perpendicular to the line $y = 2x$, $y = z + 1$, and passing at a distance 2 from O . *Ans.* $x + 2y + 2z = \pm 6$.

12. Perpendicular to the line $2y = 5x$, $z = 7x$, and passing at a distance 1 from O . *Ans.* $2x + 5y + 14z = \pm 15$.

13. Perpendicular to the line $x - 1 = y - 3 = 2z$ and passing at a distance 1 from the point $(2, 1, -3)$. *Ans.* $2x + 2y + z - 3 = \pm 3$.

14. Perpendicular to the line $x = 3y + 2 = 2z - 3$ and passing at a distance 2 from $(1, 0, 2)$. *Ans.* $6x + 2y + 3z = 12 \pm 14$.

15. Through $(1, 1, 1)$, $(2, -1, -3)$ parallel to the line $x + 2y = 6$, $y + z = 2$. *Ans.* $2x + 3y - z = 4$.

16. Through $(1, 3, 2)$, $(2, 6, -1)$ parallel to the line $7x + y - z = 7$, $x - y - z = 2$. *Ans.* $y = 3x$.

17. Through the line $x + 3y = z$, $2x + y - 2z = 1$, parallel to the line $2x = 2y = z$. *Ans.* $5x + 5y - 5z = 2$.

18. Through the line $2x + 3y - 3z = 2$, $x + y + 2z = 0$, parallel to the line $2x + 3y = 2$, $y - 2z = 1$. *Ans.* $5x + 6y + 3z = 2$.

19. Find the angle between the plane $2x - 4y + 3z = 1$ and the line $2x - 3y + 2 = 0$, $2y - z + 9 = 0$. *Ans.* $\arcsin \frac{1}{10}$.

20. Find the angle between the plane $x - y + 2z = 6$ and the line $2x - 6 = y = 2z + 3$. *Ans.* 30° .

21. Solve Ex. 31, p. 237, by a new method.

22. Solve Ex. 32, p. 237, by a new method.

23. Find the distance of the point $(3, 1, 4)$ from the line $y = x + 1$, $y + 2z = 6$. Solve in two ways. *Ans.* 3.

24. Find the distance of the origin from the line $2x - y + z = 1$, $4x - y + 3z = 3$. Solve in two ways. *Ans.* $\frac{1}{3}\sqrt{6}$.

25. Find the equations of the line through $(1, 3, 5)$ intersecting the line $\frac{x}{4} = \frac{y-1}{2} = \frac{z+3}{-1}$ at right angles. *Ans.* $\frac{x-1}{1} = \frac{y-3}{2} = \frac{z-5}{8}$.

26. Find the equations of the line through $(4, 1, -1)$ intersecting the line $\frac{x+2}{2} = \frac{y}{3} = \frac{z}{1}$ at right angles. *Ans.* $\frac{x-4}{2} = \frac{y-1}{-1} = \frac{z+1}{-1}$.

27. Solve Ex. 25 by another method.

28. Solve Ex. 26 by another method.

29. Show that the line $x - y = 2$, $z = 3$ and the plane $x - y + z = 3$ are parallel, and find the distance between them. *Ans.* $\frac{2}{3}\sqrt{3}$.

30. Show that the plane $x + 3y - z = 6$ and the line $x + y = 4$, $2y = z$ are parallel, and find the distance between them. *Ans.* $\frac{2}{11}\sqrt{11}$.

31. Find the distance between the parallel lines $x = y - 1 = -z + 4$, $x = y = -z$. *Ans.* $\sqrt{14}$.

32. Find the distance between the parallel lines $x = y + 8 = 2z - 5$, $x = y = 2z$. *Ans.* $5\sqrt{2}$.

33. Find the locus of points equidistant from the parallel lines of Ex. 31. *Ans.* $x + 2y + 3z = 7$.

34. Find the locus of points equidistant from the parallel lines of Ex. 32. *Ans.* $3x - 5y + 4z = 25$.

35. Find the shortest distance between the skew lines

$$\frac{x}{1} = \frac{y+1}{1} = \frac{z-1}{1}, \quad \frac{x}{1} = \frac{y-1}{2} = \frac{z+1}{-1}.$$

(Pass a plane through each line parallel to the other line.) *Ans.* $\frac{1}{4}\sqrt{14}$.

36. Find the shortest distance between the skew lines $x = y = 3z$, $3 - x = y = z$. (Note the suggestion in Ex. 35.) *Ans.* $\frac{3}{14}\sqrt{14}$.

CHAPTER XVIII

SURFACES

170. Symmetry. It follows from §§ 147, 150 that an equation not of the first degree represents in general a *curved surface* of some kind. We begin our study of surfaces with a discussion of symmetry.

Two points P_1, P_2 are said to be *symmetric with respect to a plane* if the plane is perpendicular to the line P_1P_2 at its midpoint—i.e. if P_2 is the image, or reflection, of P_1 in that plane. A geometric figure is symmetric with respect to a plane if corresponding to every point P_1 of the figure the image P_2 also belongs to the figure.

The definitions of *axis* and *center* of symmetry laid down in § 15 apply without change to figures in space.

The principal tests for symmetry are as follows.

THEOREM I: *A surface is symmetric with respect to the yz -plane if x can be replaced by $-x$ without changing the equation; and conversely. Similarly for symmetry with respect to the zx - and xy -planes.*

THEOREM II: *A surface is symmetric with respect to the x -axis if y and z can be replaced by $-y$ and $-z$ simultaneously without changing the equation; and conversely. Similarly for symmetry with respect to the y - and z -axes.*

THEOREM III: *A surface is symmetric with respect to the origin if x, y , and z can be replaced by $-x, -y$, and $-z$ simultaneously without changing the equation; and conversely.*

Example: The surface $yz = x^2$ is symmetric to the yz -plane, the x -axis, and the origin.

171. Sketching by parallel plane sections. One of the most effective methods of sketching a surface is by means of a series of plane sections parallel to a coördinate plane. It often happens that one particular set of sections will form a clearer picture of the surface than any other set, so that care should be taken to make the best choice.

Before making the drawing, it is well to carry out the following steps:

1. *Test the surface for symmetry.*
2. *Determine the traces on the coördinate planes.*
3. *Determine the nature of the sections parallel to each coördinate plane.*

Example: Sketch the surface $x^2 + y^2 = az$.

1. The surface is symmetric with respect to the yz - and zx -planes.

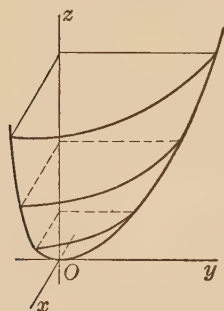
2. The traces are the parabola $y^2 = az$, the parabola $x^2 = az$, and the point-circle $x^2 + y^2 = 0$.

3. Setting $x = k$, we get as sections parallel to the yz -plane the parabolas

$$y^2 = az - k^2;$$

the sections parallel to the zx -plane are the parabolas

$$x^2 = az - k^2;$$



if a is positive, the sections parallel to the xy -plane are, when $k < 0$, imaginary; when $k > 0$, the circles

$$x^2 + y^2 = ak.$$

Evidently the surface is best pictured by means of its traces and circular sections, as in Fig. 125. On account of the symmetry, we need to draw only the portion lying in the first octant.

FIG. 125

EXERCISES

Test the surface for symmetry, find the traces, determine the nature of the sections parallel to each coördinate plane, and sketch the portion in the first octant.

- | | |
|-------------------------------|--------------------------------|
| 1. $4x^2 + 4y^2 + z^2 = 4.$ | 2. $x^2 + y^2 + 9z^2 = 36.$ |
| 3. $x^2 + 4y^2 + z^2 = a^2.$ | 4. $9x^2 + 9y^2 + 4z^2 = a^2.$ |
| 5. $4x^2 + y^2 + 3z^2 = 9.$ | 6. $x^2 + 2y^2 + 6z^2 = 12.$ |
| 7. $x^2 + z^2 = 4 - y.$ | 8. $y^2 + z^2 = 1 - x.$ |
| 9. $x^2 + 4y^2 - z^2 = 16.$ | 10. $z^2 - 2x^2 - 4y^2 = 4.$ |
| 11. $9y^2 - 4x^2 - z^2 = 36.$ | 12. $z^2 + 4y^2 - x^2 = 100.$ |
| 13. $y^2 = 4ax.$ | 14. $2xy = a^2.$ |
| 15. $x^2 + z^2 = 2ax.$ | 16. $4y^2 + z^2 = 8ay.$ |
| 17. $z^2 = x^2 + 4y^2.$ | 18. $z^2 = x^2 - 4y^2.$ |
| 19. $z = y^2 - xy.$ | 20. $z = y - xy.$ |
| 21. $z = xy.$ | 22. $z = x + yz.$ |

172. Generation of a surface by a moving curve. If a curve moves in space in any way, it evidently sweeps out, or generates, a surface of some kind. While a surface may be defined as the locus of a moving point, we more often think of it as generated by a curve moving according to a given law. In each problem, unless the contrary is indicated, the generating curve will be understood to keep the same size and shape throughout the motion.

173. Surfaces of revolution. A *surface of revolution* is a surface that can be generated by rotating a curve about a straight line. The straight line is called the *axis of revolution*, and the revolved curve is the *generating curve*, or *generator*. The sections by planes through the axis are *meridian sections*, or simply *meridians*; those by planes perpendicular to the axis are *right sections*, or *parallels*. Evidently the right sections are circles with centers in the axis of revolution.

The more important surfaces of revolution are as follows.

<i>Surface</i>	<i>Generated by the rotation of</i>
The <i>sphere</i>	A circle about a diameter.
The <i>prolate spheroid</i>	An ellipse about its major axis.
The <i>oblate spheroid</i>	An ellipse about its minor axis.
The <i>hyperboloid of revolution of one sheet</i>	A hyperbola about its conjugate axis.
The <i>hyperboloid of revolution of two sheets</i>	A hyperbola about its transverse axis.
The <i>paraboloid of revolution</i>	A parabola about its axis.
The <i>circular cylinder</i>	A straight line about a line parallel to it.
The <i>circular cone</i>	A straight line about a line intersecting it obliquely.
The <i>torus</i>	A circle about a line in its plane, not intersecting it.

The spheroids are also called *ellipsoids of revolution*. The equations of all of these surfaces, with the exception of the torus, are of the second degree.

In the simpler cases, the equation of a surface of revolution is easily derived when the axis and generating curve are given. The method is best explained by means of

Examples: (a) Derive the equation of the cone generated by revolving the line

$$(1) \quad x + y = 1, \quad z = 0$$

about Ox (the line AQB in Fig. 126).

In the rotation, every point of the generating line describes a circle parallel to the yz -plane, with its center in the x -axis. Let Q be any point of the generating line, and P any point of the circle described by Q , so that P is a

general point on the surface. If the coördinates of P be denoted by (x, y, z) , those of Q are $(x, y_Q, 0)$, where y_Q denotes the y -coördinate of Q , viz. the distance LQ . We have to obtain an equation involving x, y , and z but not containing y_Q .

The equation of the circular path of Q in its own plane is

$$(2) \quad y^2 + z^2 = y_Q^2.$$

But Q lies on the generating line (1), so that its coördinates $(x, y_Q, 0)$ may be substituted:

$$x + y_Q = 1,$$

or

$$(3) \quad y_Q = 1 - x.$$

Eliminating y_Q between (2) and (3), we get

$$y^2 + z^2 = (1 - x)^2,$$

which is the equation of the cone. As a check we note that the traces are, in the xy -plane, the given line.

$$y = 1 - x$$

and the line

$$y = -(1 - x),$$

in the zx -plane the lines

$$z = \pm (1 - x),$$

and in the yz -plane the circle

$$y^2 + z^2 = 1,$$

all of which are seen by Fig. 126 to be correct.

It might be pointed out that equations (2) and (3) together constitute parametric equations of the cone in terms of y_Q (§ 111).

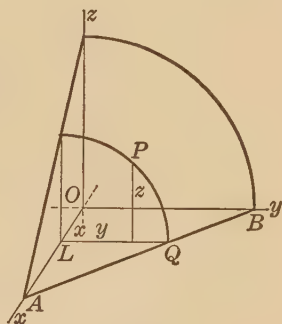


FIG. 126

(b) Derive the equation of the oblate spheroid.

We will generate the spheroid by revolving the ellipse

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1, \quad x = 0$$

(the ellipse AQB in Fig. 127) about its minor axis Oz . Let $P : (x, y, z)$ be any point of the surface, and $Q : (0, y_Q, z)$ the corresponding point of the generating ellipse. The equation of the circle described by Q is

$$(4) \quad x^2 + y^2 = y_Q^2.$$

Since Q is on the ellipse, we have

$$(5) \quad \frac{y_Q^2}{a^2} + \frac{z^2}{b^2} = 1.$$

Substituting the value of y_Q^2 from (4) in (5), we get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1.$$

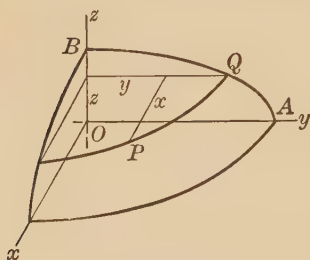


FIG. 127

This result may be checked as in the previous example.

EXERCISES

Derive the equations of the surfaces generated as indicated, drawing the figure in each case. Check by means of the traces.

- By revolving the line $y = 3x, z = 0$ (a) about Ox ; (b) about Oy .
Ans. (a) $9x^2 - y^2 - z^2 = 0$; (b) $9x^2 - y^2 + 9z^2 = 0$.
- By revolving the line $z = 2y, x = 0$ (a) about Oy ; (b) about Oz .
Ans. (a) $x^2 + z^2 = 4y^2$; (b) $4x^2 + 4y^2 = z^2$.
- By rotating the line $z = x + 3, y = 0$ (a) about Ox ; (b) about Oz .
- By rotating the line $y = 3z + 1, x = 0$ (a) about Oy ; (b) about Oz .
- By rotating the circle $x^2 + y^2 = a^2, z = 0$ about Oy .
- By rotating the circle $x^2 + z^2 = 2ax, y = 0$ about Ox .
- By revolving the parabola $y^2 = 4az, x = 0$ (a) about its axis; (b) about the tangent at the vertex.
Ans. (b) $y^4 = 16a^2(x^2 + z^2)$.
- By revolving the parabola $y^2 = 4x - 4, z = 0$ (a) about its axis; (b) about its directrix.
Ans. (b) $y^4 - 16x^2 + 8y^2 - 16z^2 + 16 = 0$.

9. By revolving the ellipse $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$, $y = 0$ about each of its axes in turn.

10. By revolving a hyperbola about each of its axes in turn.

11. By rotating the line $x = 3$, $z = 0$ about Oy .

12. By rotating the line $z = 5$, $y = 0$ about Ox .

13. By revolving the curve $z = y^3$, $x = 0$ (a) about Oy ; (b) about Oz .

14. By revolving the curve $xy^2 = 1$, $z = 0$ (a) about Ox ; (b) about Oy .

15. By revolving the line $x = y = 3$ about Oz .

16. By revolving the line $y = 3$, $z = 4$ about Ox .

17. By revolving the line $z = x + 4$, $y = 4$ about the line $y = 4$, $z = 0$.
Ans. $(y - 4)^2 + z^2 = (x + 4)^2$.

18. By revolving the circle $(x - 4)^2 + z^2 = 4$, $y = 2$ about the line $x = 4$, $y = 2$.
Ans. $(x - 4)^2 + (y - 2)^2 + z^2 = 4$.

19. By revolving the parabola $z^2 = y - 1$, $x = 2$ about its axis.

Ans. $(x - 2)^2 + z^2 = y - 1$.

20. By revolving the hyperbola $(x - 2)^2 - (z - 2)^2 = 1$, $y = 4$ about each of its axes in turn.

21. By revolving the line $x = 3$, $z = 1$ about the line $x = 4$, $z = 3$.

22. By revolving the line $x = 5$, $y = 3$ about the line $x = 3$, $y = 4$.

23. By revolving the line $x = 3y$, $y = z$ (a) about Ox ; (b) about Oy ; (c) about Oz .
Ans. (a) $9y^2 + 9z^2 = 2x^2$.

24. By revolving the line through $(0, 0, 0)$ and $(1, 2, 3)$ about each of the axes in turn.
Ans. About Oy , $2x^2 + 2z^2 = 5y^2$.

25. By revolving the line $y = z$, $x = a$ about Oz .

Ans. $x^2 + y^2 - z^2 = a^2$.

26. By revolving the line $x + y = a$, $z = a$ about Oy .

Ans. $x^2 + z^2 - y^2 = 2a(a - y)$.

27. By revolving the line through $(1, 0, 0)$ and $(0, 1, 2)$ about Ox , Oy , Oz in turn.
Ans. About Oy , $x^2 + z^2 = 5y^2 - 2y + 1$.

28. By revolving the line through $(0, 2, 0)$ and $(1, 1, 1)$ about Ox , Oy , Oz in turn.
Ans. About Ox , $y^2 + z^2 = 2x^2 - 4x + 4$.

29. By revolving the circle $y^2 + z^2 = 2ay$, $x = 0$ about Oz .

Ans. $(x^2 + y^2 + z^2)^2 = 4a^2(x^2 + y^2)$.

30. By revolving the circle $x^2 + (z - 2)^2 = 1$, $y = 0$ about Ox .

Ans. $(x^2 + y^2 + z^2 - 5)^2 = 16(1 - x^2)$.

31. By revolving the circle $(x - b)^2 + y^2 = a^2$, $z = 0$ about Oy .

Ans. $(x^2 + y^2 + z^2 - a^2 - b^2)^2 = 4b^2(a^2 - y^2)$.

174. The sphere. A *sphere* is the locus of a moving point that remains at a constant distance from a fixed point. The constant distance is the *radius* and the fixed point the *center*.

The equation of a sphere of given center and radius can be written down at once by (1), § 138. The equation of a sphere of radius a is, if the center is the origin,

$$(1) \quad x^2 + y^2 + z^2 = a^2;$$

if the center is the point (h, k, l) ,

$$(2) \quad (x - h)^2 + (y - k)^2 + (z - l)^2 = a^2.$$

Every equation of the form

$$(3) \quad Ax^2 + Ay^2 + Az^2 + Gx + Hy + Iz + K = 0$$

can be reduced to the form (2) by completing the squares in x , y , and z ; hence *every equation of the form (3) represents a sphere* (exceptionally, a point, or no locus—cf. § 37).

EXERCISES

Find the center and radius of each of the following spheres.

1. $x^2 + y^2 + z^2 - 4x + 2y + 6z - 2 = 0$.
2. $x^2 + y^2 + z^2 + 3x - y - 2z = 2$.
3. $x^2 + y^2 + z^2 = 4x - 2y$.
4. $4x^2 + 4y^2 + 4z^2 = z$.
5. $3x^2 + 3y^2 + 3z^2 - x - 2y + 4z - 2 = 0$.
6. $2x^2 + 2y^2 + 2z^2 - 3x + 5z + 1 = 0$.

Write the equations of the following spheres.

7. Of radius 2 with center at $(1, -3, -5)$.
8. Of radius 3 with center at $(2, 0, -3)$.
9. With center at O tangent to the plane $3x + y - 3z + 2 = 0$.
10. With center at O tangent to the plane $9x - 2y + 6z + 11 = 0$.
11. With center at $(2, 1, -3)$ tangent to the plane $x + y - z = 3$.
12. With center at $(0, 3, -5)$ tangent to the plane $2x - 3y - z = 3$.
13. With center on the line $x = y = z$ and passing through $(4, 2, 3)$, $(-1, -2, 2)$.
Ans. $(x - 1)^2 + (y - 1)^2 + (z - 1)^2 = 14$.

14. Find the tangent plane to the sphere $x^2 + y^2 + z^2 = 6$ at $(1, 1, -2)$. *Ans.* $x + y - 2z = 6$.

15. Find the tangent plane to the sphere of Ex. 13 at $(4, 2, 3)$.

16. Prove that the line $3x - y = 2$, $2x - z = 4$ touches the sphere $x^2 + y^2 + z^2 = 6$ at $(1, 1, -2)$. (Cf. Ex. 14.)

17. Solve Ex. 16 by a second method.

18. Solve Ex. 16 by a third method.

19. Prove that the line $y + z = 0$, $x + 2y + 5 = 0$ touches the sphere of Ex. 13 at $(-1, -2, 2)$. (Cf. Ex. 15.)

20. Solve Ex. 19 by a second method.

21. Solve Ex. 19 by a third method.

22. Find the tangent plane at $(1, -1, 1)$ to the sphere $2x^2 + 2y^2 + 2z^2 - 3x - y + z - 5 = 0$. *Ans.* $x - 5y + 5z = 11$.

23. Prove that a sphere is determined by four points.

24. Find the equation of the sphere through the points $(0, 0, 0)$, $(-1, 0, 0)$, $(0, 4, 0)$, $(1, 2, -1)$. *Ans.* $x^2 + y^2 + z^2 + x - 4y - z = 0$.

25. Find the equation of a sphere of radius 3 touching the plane $4x + y + 8z = 9$ at $(1, 1, \frac{1}{2})$.

26. Find the equation of a sphere touching the plane $3x + y + z = 6$ at $(1, 0, 3)$ and passing through $(2, 2, 1)$.

$$\text{Ans. } (x - \frac{1}{2})^2 + (y - \frac{3}{2})^2 + (z - \frac{9}{2})^2 = \frac{9}{4}.$$

27. Find the equation of the sphere touching the plane $x + y + 2z = 4$ at $(2, 0, 1)$ and passing through $(0, 1, 0)$.

$$\text{Ans. } (x - 1)^2 + (y + 1)^2 + (z + 1)^2 = 6.$$

175. Quadric surfaces. A surface whose equation is of the second degree is called a *quadric surface*. The quadrics occupy much the same place among surfaces that the conics hold among curves: next to the plane, they are by far the most important class of surfaces.

The quadrics proper are of nine species: the ellipsoid (of which the sphere is a special case), the hyperboloids (two species), the paraboloids (two species), the quadric cylinder (three species), and the quadric cone. In addition there are the degenerate forms, analogous to those of Chap. VI—two parallel, coincident, or intersecting planes, a single straight line, or a point.

176. The ellipsoid. The locus of the equation

$$(1) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is called an *ellipsoid*. This surface is symmetric with respect to all three coördinate planes and lies entirely within the limits $-a \leq x \leq a$, $-b \leq y \leq b$, $-c \leq z \leq c$.

The segments of length $2a$, $2b$, $2c$ cut off on the coördinate axes are the *axes* of the ellipsoid, the point O is the *center*. It will appear later (Ex. 18, p. 273) that every plane section of an ellipsoid is an ellipse.

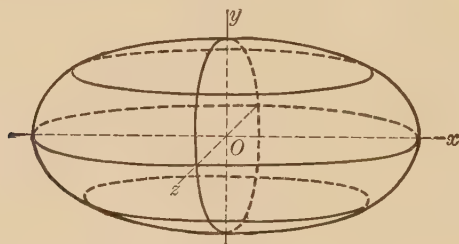


FIG. 128

When two of the semiaxes a , b , c are equal, so that the sections by planes perpendicular to the third axis are circles, the surface becomes an *ellipsoid of revolution*: if the equal axes are shorter than the third, a *prolate spheroid*, if longer, an *oblate spheroid*. When $a = b = c$, the surface is a sphere.

177. The hyperboloid of one sheet. The equation

$$(1) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

represents a *hyperboloid of one sheet* (Fig. 129). The sections parallel to the yz - and zx -planes are hyperbolas; parallel to the xy -plane, ellipses. The intercepts on Ox are $\pm a$; on Oy , $\pm b$; on Oz , imaginary. The surface is a con-

nected, open surface extending to infinity in both directions along the z -axis.

If $a = b$, the elliptic sections become circular and the surface is the *hyperboloid of revolution of one sheet*. If $a = c$ (or $b = c$), but $a \neq b$, the surface is not one of revolution: the result merely means that the sections $y = k$ (or $x = k$) are equilateral hyperbolas.

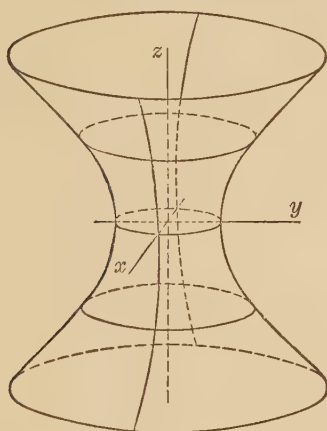


FIG. 129

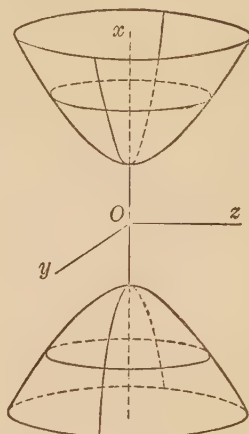


FIG. 130

178. The hyperboloid of two sheets. The equation

$$(1) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

represents a *hyperboloid of two sheets* (Fig. 130). A study of the sections shows that the surface consists of two disconnected sheets, one in the region $x > a$, the other in the region $x < -a$, each opening out larger along Ox as x increases numerically. Note the unconventional arrangement of the axes in Fig. 130.

When $b = c$, the elliptic sections become circles, and the surface becomes the *hyperboloid of revolution of two sheets*.

179. Classification. From the results of §§ 176–178 we conclude at once:

Every equation of the form

$$Ax^2 + By^2 + Cz^2 = 1,$$

where A, B, C are all different from 0, represents:

- (a) if all signs are positive, an ellipsoid;
- (b) if one sign is negative, a hyperboloid of one sheet;
- (c) if two signs are negative, a hyperboloid of two sheets.

EXERCISES

Classify the surface, study the sections parallel to the coördinate planes, state whether the surface is one of revolution, and make a sketch.

- | | |
|---|---|
| 1. $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{1} = 1.$ | 2. $\frac{x^2}{1} + \frac{y^2}{2} + \frac{z^2}{4} = 1.$ |
| 3. $\frac{x^2}{9} + \frac{y^2}{1} - \frac{z^2}{9} = 1.$ | 4. $\frac{z^2}{4} + \frac{x^2}{3} - \frac{y^2}{4} = 1.$ |
| 5. $\frac{x^2}{4} - \frac{y^2}{1} - \frac{z^2}{4} = 1.$ | 6. $\frac{z^2}{9} - \frac{x^2}{2} - \frac{y^2}{1} = 1.$ |
| 7. $3x^2 + 3y^2 + 4z^2 = 12.$ | 8. $x^2 - 4y^2 - 4z^2 = 1.$ |
| 9. $2y^2 - 3z^2 - x^2 = 6.$ | 10. $4x^2 + 4y^2 + 4z^2 = 1.$ |
| 11. $4y^2 + 4z^2 - 2x^2 = 1.$ | 12. $x^2 + 4y^2 + z^2 = 9.$ |
| 13. $x^2 - y^2 + 2z^2 + 4 = 0.$ | 14. $z^2 - 2x^2 - 4y^2 + 4 = 0.$ |
| 15. $x^2 - y^2 + z^2 = 1.$ | 16. $x^2 - y^2 + z^2 + 1 = 0.$ |
| 17. $2x^2 + 2y^2 + z^2 = 4.$ | 18. $3x^2 + y^2 + 3z^2 = 3.$ |
| 19. $x^2 + 4y^2 + 2z^2 = 0.$ | 20. $2x^2 + y^2 + z^2 + 2 = 0.$ |

By completing the squares in x, y, z (cf. the methods employed in Chap. VI), show that the equation represents an ellipsoid or hyperboloid with axes parallel to the coördinate axes, locate the center, and classify the surface.

- 21. $x^2 - y^2 - z^2 + 4x - 8y + 2z - 17 = 0.$
- 22. $4x^2 + y^2 - 2z^2 + 6y + 4z + 4 = 0.$
- 23. $3x^2 + y^2 + 2z^2 + 3x + 3y + 4z = 0.$
- 24. $x^2 + 4y^2 + z^2 - 4x - 8y + 8z + 15 = 0.$
- 25. $2x^2 - 3y^2 - 4z^2 - 12x - 6y + 16 = 0.$
- 26. $x^2 - 4y^2 + 3z^2 + 2x + 16y + 6z = 0.$

27. $6x^2 - 3y^2 + 4z^2 = 12y.$

28. $x^2 - y^2 - 8z^2 = 2y + 8z.$

29. The hyperboloid of revolution of one sheet may be defined as the surface generated by revolving one of two skew lines about the other. Prove this statement by revolving the line $y = h, z = mx$ about Ox . Is the proof general? *Ans.* $y^2 + z^2 - m^2x^2 = h^2.$

30. Find the equation of the hyperboloid generated by revolving the line $y = x, z = 2$ about Ox . *Ans.* $y^2 + z^2 - x^2 = 4.$

31. Find the equation of the hyperboloid generated by revolving the line $y = 3, z = 2x$ about Oz .

32. Find the equation of the hyperboloid generated by revolving the line $x = 4, 3y - 2z = 0$ about Oy .

33. A point moves so that the sum of its distances from two fixed points is constant. Show that its locus is a prolate spheroid. Take the fixed points as $(\pm c, 0, 0)$.

34. A point moves so that the difference of its distances from two fixed points is constant. Show that its locus is a hyperboloid of revolution of two sheets.

180. The elliptic paraboloid. The locus of the equation

$$(1) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

is called an *elliptic paraboloid* (Fig. 131).

From equation (1) we find that the elliptic paraboloid has two planes of symmetry; it also has one line of symmetry, called the *axis* of the surface. The axis intersects the surface in a single point, called the *vertex*. The surface lies entirely on one side of the xy -plane, and extends to infinity along Oz , opening upward or downward according as c is positive or negative.

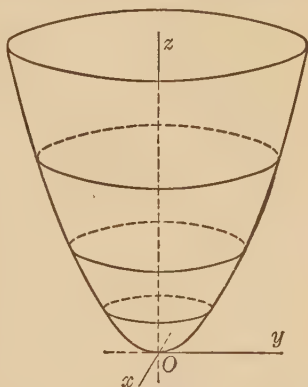


FIG. 131

When $a = b$, the surface is the *paraboloid of revolution*.

181. The hyperbolic paraboloid. The surface

$$(1) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

is called a *hyperbolic paraboloid*. The sections $y = k$ are parabolas opening upward or downward, according to the sign of c ; sections $x = k$ are parabolas opening in the opposite direction. It thus appears that the surface is "saddle-shaped," as shown in Fig. 132.

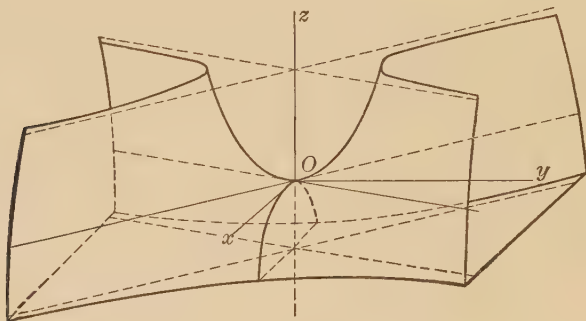


FIG. 132

The hyperbolic paraboloid cannot in any case be a surface of revolution. The case $a = b$ is merely the case in which the sections $z = k$ are equilateral hyperbolas.

EXERCISES

Classify the surface, study the sections parallel to the coordinate planes, state whether the surface is one of revolution, and make a sketch.

- | | |
|-----------------------------------|----------------------------------|
| 1. $2x^2 + z^2 - 4y = 0$. | 2. $x^2 + 2y^2 - 6z = 0$. |
| 3. $x^2 - y^2 + 4z = 0$. | 4. $x^2 - y^2 - z = 0$. |
| 5. $x^2 + 2y^2 + 6z^2 = 6$. | 6. $y^2 - 2z^2 - 2x^2 = 2$. |
| 7. $2x^2 + 2z^2 = y$. | 8. $4x^2 - 4z^2 = y$. |
| 9. $4x + y^2 - 4z^2 = 0$. | 10. $8x - y^2 - 4z^2 = 0$. |
| 11. $x^2 + 4y^2 - 4z^2 + 9 = 0$. | 12. $x^2 + y^2 + 4z^2 = 16$. |
| 13. $8x + y^2 + 2z^2 = 0$. | 14. $4x^2 + y + 4z^2 = 0$. |
| 15. $x^2 + y^2 + z^2 = 8x$. | 16. $x^2 - y^2 + 2z^2 + 2 = 0$. |

By completing squares, locate the vertex and classify the surface.

17. $y^2 - 4z^2 + 2x + 2y + 8z = 0$. Ans. $V: (-\frac{3}{2}, -1, 1)$.

18. $x^2 - 2z^2 + 4x - 6y - 2 = 0$. Ans. $V: (-2, -1, 0)$.

19. $2x^2 + 2z^2 + 4x + 3y + 4z = 3$.

20. $4y^2 + z^2 + 8x - 8y + 4z + 2 = 0$.

21. In Fig. 132, is the constant c positive or negative?

22. Find the equation of a paraboloid with vertex at O , axis Oy , and passing through $(1, -2, 1)$, $(-3, -3, 2)$. Ans. $x^2 - 3z^2 = y$.

23. Find the equation of a paraboloid with vertex at O , axis Ox , and passing through $(3, 1, 2)$, $(3, 3, 0)$. Ans. $y^2 + 2z^2 = 3x$.

182. Cylinders. A *cylinder* is the surface described by a moving line which remains parallel to its original position and always intersects a fixed curve, called the *directing curve*. Thus the cylinder is completely covered by straight lines, called *generators*, all of which are parallel.

The section by any plane perpendicular to the generators is called a *right section*; it is obvious that all right sections, and in fact all parallel plane sections, are equal curves. If the right section has a center, the line through this center parallel to the generators is the *axis* of the cylinder.

183. Equations in two variables: cylinders perpendicular to a coördinate plane. Consider an

Example: Determine the surface represented by the equation (Fig. 133)

$$(1) \quad x^2 + y^2 = a^2.$$

The xy -trace of this surface is a circle. At any point M of this circle erect a perpendicular to the xy -plane, and let $P: (x, y, z)$ be any point of the perpendicular. Now the coördinates of M satisfy (1), and those of P must do the same, since its x - and y -coördinates are the same as those of M , and z does not occur in the equation. Thus P is on

the surface, and since P is any point of the perpendicular at M , that entire line must lie in the surface. Further, since M is any point of the circle, the surface includes all

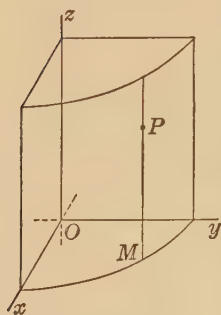


FIG. 133

lines perpendicular to the xy -plane through points of the circle. On the other hand, if a point does not lie in one of these lines its coördinates obviously cannot satisfy the equation, since its x - and y -coördinates cannot be identical with those of any point of the circle. Thus the equation represents a *cylinder with generators perpendicular to the xy -plane*, the directing curve being the curve represented by the given equation

in that plane.

Evidently a strictly analogous result holds for any equation in two variables. Conversely, it is easily seen that the equation of every cylinder with generators perpendicular to a coördinate plane involves only two variables. Thus we have the

THEOREM: *An equation in two variables represents a cylinder whose generators are perpendicular to the plane of the two variables and whose directing curve is the curve represented by the given equation in that plane; and conversely.*

COROLLARY: *A cylinder is a quadric surface if and only if its right section is a conic.*

The truth of the corollary appears at once from the fact that, when the generators are perpendicular to a coördinate plane, the equation of the cylinder is identical with that of the directing curve.

A quadric cylinder is called *elliptic*, *parabolic*, or *hyperbolic*, according to the nature of its right section.

184. Cones. A *cone* is the surface generated by a moving line that always passes through a fixed point, called the *vertex*, and intersects a fixed curve, called the *directing curve*. Thus the surface is completely covered by straight lines, or *generators*, all passing through a fixed point. Like the cylinder, a cone may or may not be a quadric surface.

185. The elliptic cone. The locus of the equation

$$(1) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

is a *quadric cone*. This surface evidently has three planes and three axes of symmetry, and a center, or *vertex*. A study of the sections shows that the surface consists of two open sheets extending to infinity along the z -axis.

By suitable coördinate transformations (analogous to those of §§ 62–63), the equation of every quadric cone can be reduced to the form (1). Hence there is only a single species of quadric cone; from this one surface every kind of conic can be cut. Since the surface is most clearly visualized by means of its elliptic sections, it is usually called the *elliptic cone*.

The line through the centers of the elliptic sections is called the *axis* of the cone, and sections by planes perpendicular to the axis are *right sections*.

When $a = b$, the right sections are circles, and the surface is the *circular cone*, or *cone of revolution*.

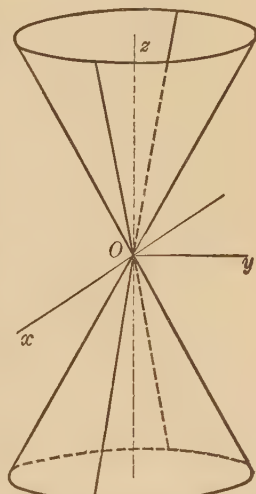


FIG. 134

EXERCISES

Draw the following cylinders.

1. $z^2 = 4ay.$

2. $x^2 - 4z^2 = 16.$

3. $x^2 + y^2 = 4ay.$

4. $2xy = a^2.$

5. $z = x^2 - 2x.$

6. $2y^2 + z^2 = 4y.$

7. $y^2 - z^2 = 2y.$

8. $x^2 - 2y = 2.$

Draw the following cones.

9. $4x^2 - 9y^2 + 36z^2 = 0.$

10. $4x^2 - y^2 - z^2 = 0.$

11. $x^2 + y^2 = 9z^2.$

12. $3x^2 + y^2 - 3z^2 = 0.$

13. $9x^2 - 4y^2 = 9z^2.$

14. $2x^2 - 3y^2 + 4z^2 = 0.$

By completing squares, show that the given equation represents a cone; find its vertex and axis.

15. $x^2 + y^2 - z^2 - 4x + 2y + 2z - 4 = 0.$ Ans. V: (2, -1, 1).

16. $x^2 - 2y^2 - z^2 + 2x + 12y - 4z - 21 = 0.$

Ans. V: (-1, 3, -2).

17. $3x^2 - 2y^2 - z^2 - 12x - 2z + 11 = 0.$

18. $2x^2 - y^2 + 2z^2 + 4y + 4z - 2 = 0.$

19. A point moves so as to remain equidistant from the x -axis and the plane $y + z = a$. Find the equation of its locus. What kind of surface is generated? Ans. $2y^2 + 2z^2 = (y + z - a)^2.$

20. A point lies at the distance a from the line $x = a, z = 2a$. Find the equation of its locus.

21. A moving point is equidistant from the point $(a, 0, 0)$ and the line $x + a = 0, y = 0$. Find the equation of its locus. Ans. $z^2 = 4ax.$

22. A fixed line and fixed plane are perpendicular to each other; a point moves so that its distance from the line bears a constant ratio to its distance from the plane. Find the equation of its locus. (Take the line as x -axis, the plane as yz -plane.)

186. Ruled surfaces. A surface that can be generated by a moving straight line is called a *ruled surface*. It follows from the definition that such a surface is completely covered by straight lines, and this property might equally well be used as a definition. The straight lines are called *generators*, or *rulings*.

The following facts appear from the definitions of the surfaces in question:

*A cylinder is a ruled surface all of whose generators are parallel; and conversely.**

*A cone is a ruled surface all of whose generators are concurrent; and conversely.**

Examples: (a) Prove that the equation

$$(1) \quad y^2 + (x + z)^2 = a^2$$

represents a cylinder.

Setting $y = k$ in (1), we get $x + z = \pm \sqrt{a^2 - k^2}$. Thus, for values of $k \leq a$, our surface contains the family of straight lines

$$y = k, \\ x + z = \pm \sqrt{a^2 - k^2}$$

and is a ruled surface; since these lines are all parallel, it is a cylinder. The xy - and yz -traces are circles; the portion of the surface lying in the first octant is shown in Fig. 135

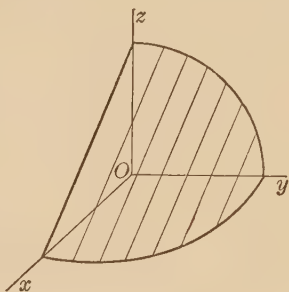


FIG. 135

(b) By discovering a family of concurrent generators, prove directly that the equation

$$x^2 + y^2 - z^2 = 0$$

represents a cone.

Writing the equation in the form

$$(2) \quad z^2 - x^2 = y^2,$$

we see that the lines

$$(3) \quad z + x = ky, \quad k(z - x) = y$$

lie in the surface. For, when k is eliminated from equations (3), equation (2) results. And all the lines (3) pass through the origin.

* The converse theorems are true only if we exclude, or admit as special cases, a plane or group of planes, a straight line (circular cylinder of radius 0), etc.

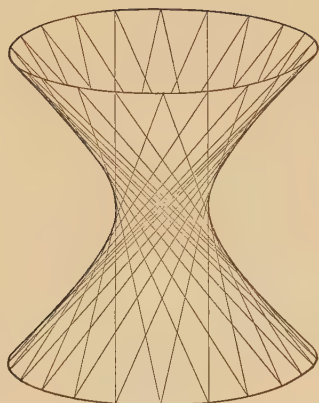


FIG. 136

187. Other ruled quadrics.

Aside from the cylinders and the cone, the only ruled quadrics are the hyperboloid of one sheet and the hyperbolic paraboloid.

Let us write the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

in the form

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2},$$

and this in turn in the form

$$(1) \quad \left(\frac{y}{b} + \frac{z}{c}\right)\left(\frac{y}{b} - \frac{z}{c}\right) = \left(1 + \frac{x}{a}\right)\left(1 - \frac{x}{a}\right).$$

Consider now the family of straight lines represented by the two equations

$$(2) \quad k\left(\frac{y}{b} + \frac{z}{c}\right) = 1 + \frac{x}{a}, \quad \frac{y}{b} - \frac{z}{c} = k\left(1 - \frac{x}{a}\right),$$

where k is an arbitrary constant. Upon multiplying both members of (1) by k , it is easily seen that that equation is satisfied whenever equations (2) are both satisfied, and conversely, so that the hyperboloid contains, and is entirely covered by, this family of lines.

Likewise the hyperboloid contains, and is completely covered by, the family of lines

$$k'\left(\frac{y}{b} - \frac{z}{c}\right) = 1 + \frac{x}{a}, \quad \frac{y}{b} + \frac{z}{c} = k'\left(1 - \frac{x}{a}\right).$$

Combining this result with that of Ex. 15 below (which may be solved by similar reasoning), we have

THEOREM I: *The hyperboloid of one sheet, and the hyperbolic paraboloid, are ruled surfaces containing two distinct families of rectilinear generators.*

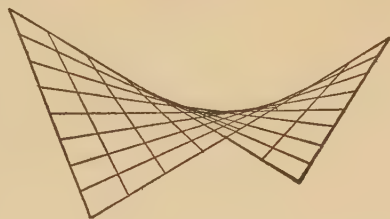


FIG. 137

The following theorem, expressing a characteristic property of the hyperbolic paraboloid, may be proved by the student (Exs. 16–18 below).

THEOREM II: *The hyperbolic paraboloid may be generated by a straight line moving parallel to a fixed plane and intersecting two fixed skew lines which themselves intersect the fixed plane.*

EXERCISES

By discovering a set of parallel generators, prove that the surface is a cylinder; sketch the portion in the first octant.

1. $(y + z)^2 = 1 - x.$

2. $(x - y)^2 = 1 - z.$

3. $x(y + z) = a^2.$

4. $y^2 = 2y - x - z.$

5. $x^3 = 2x - y - z.$

6. $(x + y)^3 = 1 - z.$

7. $y^3 = a(x + z)^2.$

8. $x(y + z)^2 = a^3.$

By discovering a set of concurrent generators, prove that the surface is a cone; sketch the portion in the first octant. (To sketch, take a trace in any convenient plane parallel to a coördinate plane, and draw lines from the vertex to this trace.)

9. $y^2 = 4zx.$

10. $xy = y^2 + z^2.$

11. $x^2 - y^2 + z^2 = 2xy.$

12. $x^2 = 2zx - yz.$

13. $y^3 = x^2z.$

14. $y^3 - x^3 = y^2z.$

15. Prove that the hyperbolic paraboloid (1), § 181, contains the two families of lines

$$k\left(\frac{x}{a} + \frac{y}{b}\right) = \frac{z}{c}, \quad \frac{x}{a} - \frac{y}{b} = k \quad \text{and} \quad k'\left(\frac{x}{a} - \frac{y}{b}\right) = \frac{z}{c}, \quad \frac{x}{a} + \frac{y}{b} = k'.$$

For the first family of lines in Ex. 15, prove the following statements.

16. All the lines are parallel to the fixed plane $bx - ay = 0$.

17. All the lines intersect the fixed line $\frac{x}{a} + \frac{y}{b} = 0, z = 0$. Find the point of intersection. *Ans.* $(\frac{1}{2}ak, -\frac{1}{2}bk, 0)$.

18. All the lines intersect the fixed line $\frac{x}{a} - \frac{y}{b} = \frac{z}{c}, \frac{x}{a} + \frac{y}{b} = 1$. Find the point of intersection.

Draw the following hyperbolic paraboloids.

19. $az = xy$.

20. $(x - a)(y - b) = bz$.

21. $zx = (y - z)(z - a)$.

22. $(x + z - 2a)(a - z) = y(z - 2a)$.

23. A *conoid* is the surface generated by a line moving always parallel to a given plane and intersecting a fixed line and a fixed curve. Show that the equation $a^2x^2 = (a^2 - z^2)(a - y)^2$ represents a conoid, and draw the surface.

24. Draw the conoid $x^2y^2 = (y - z)^2(a^2 - y^2)$.

188. Ruled surface generated in a given manner. It is usually a simple matter to find the equation of a ruled surface when the law of motion of the generating line is given.

Examples: (a) Find the equation of the cylinder whose directing curve is the parabola

$$z^2 = 4y, \quad x = 0,$$

and whose generators have direction components 1, 1, -1.

Let P be any point of the surface, and $Q : (0, y_Q, z_Q)$ the point in which the generator through P intersects the directing curve. The equations of the line QP are

$$\frac{x}{1} = \frac{y - y_Q}{1} = \frac{z - z_Q}{-1},$$

whence .

$$y_Q = y - x, \quad z_Q = z + x.$$

Further,

$$z_Q^2 = 4y_Q.$$

Substituting, we find

$$(z + x)^2 = 4(y - x).$$

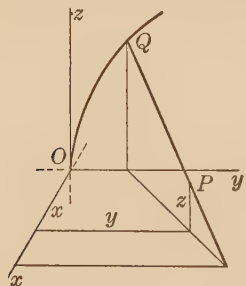


FIG. 138

(b) Find the equation of the cone with vertex at V : $(0, 0, 1)$ and directing curve

$$x^2 + y^2 = 1, \quad z = 0.$$

Let $P : (x, y, z)$ be any point of the surface, and $Q : (x_Q, y_Q, 0)$ the point where the generator through P intersects the directing curve. The equations of the generator VQ in the symmetric form are

$$\frac{x}{x_Q} = \frac{y}{y_Q} = \frac{z - 1}{-1},$$

whence

$$x_Q = \frac{-x}{z - 1}, \quad y_Q = \frac{-y}{z - 1}.$$

But

$$x_Q^2 + y_Q^2 = 1.$$

Eliminating x_Q and y_Q , we find

$$x^2 + y^2 = (z - 1)^2.$$

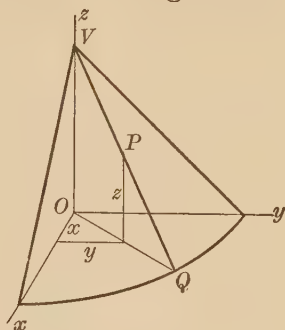


FIG. 139

(c) Find the equation of the hyperbolic paraboloid generated by a line moving parallel to the xy -plane, intersecting the lines $x + z = a$, $y = 0$ and $y = b$, $x = 0$.

Let $P : (x, y, z)$ be any point of the surface, and QR the generating line through P . The equation of that line referred to NQ , NR as axes is, in the intercept form,

$$\frac{x}{x_Q} + \frac{y}{y_R} = 1.$$

But

$$x_Q + z = a, \quad y_R = b,$$

whence

$$\frac{x}{a - z} + \frac{y}{b} = 1,$$

or

$$yz - bx - ay - bz + ab = 0.$$

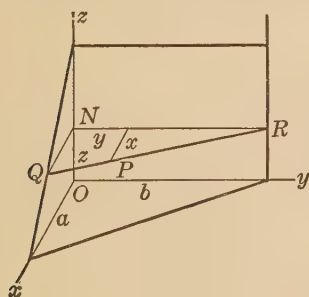


FIG. 140

EXERCISES

Find the equations of the following cylinders.

1. Directing curve $x^2 + y^2 = 1$, $z = 0$; generators parallel to the radius vector of the point $(1, 2, 3)$. *Ans.* $(3x - z)^2 + (3y - 2z)^2 = 9$.

2. Directing curve $x^2 - z^2 = a^2$, $y = 0$; generators parallel to the radius vector of the point (a, a, a) . *Ans.* $x^2 - z^2 - 2xy + 2yz = a^2$.

3. Directing curve $y^2 = z$, $x = 0$; generators perpendicular to the plane $2x - 3y + z = 4$. *Ans.* $9x^2 + 12xy + 4y^2 + 2y - 4z = 0$.

4. Directing curve $y = x^3$, $z = 0$; generators perpendicular to the plane $x + 2y + z = 1$. *Ans.* $(x - z)^3 = y - 2z$.

5. Directing curve $x^2 = 1 - z$, $y = 0$; having the line $x - y = 1$, $z = 0$ as one of its generators. *Ans.* $(x - y)^2 = 1 - z$.

6. Directing curve $2x^2 + z^2 = 2a^2$, $y = 0$; having the line $z = y$, $x = a$ as one of its generators. Show that this is a circular cylinder. *Ans.* $2x^2 + (z - y)^2 = 2a^2$.

Find the equations of the following cones.

7. Vertex $(0, 0, 0)$; directing curve $y^2 - z^2 = a^2$, $x = a$.

8. Vertex $(0, 0, 0)$; directing curve $y^2 = 8x$, $z = 2$. *Ans.* $y^2 = 4zx$.

9. Vertex $(0, 4, 0)$; directing curve $9x^2 + 4z^2 = 36$, $y = 0$.

10. Vertex $(a, 0, 0)$; directing curve $y^2 + z^2 = 2ay$, $x = 0$. *Ans.* $y^2 + z^2 + 2xy = 2ay$.

11. Vertex $(1, 3, 5)$; directing curve $x^2 + y^2 = 4$, $z = 0$. *Ans.* $(5x - z)^2 + (5y - 3z)^2 = 4(z - 5)^2$.

12. Vertex $(0, 0, 0)$; directing curve $y = x^4$, $z = 2$. *Ans.* $8x^4 = yz^3$.

Find the equations of the following hyperbolic paraboloids.

13. Generated by a line parallel to the xy -plane intersecting the lines $z = x$, $y = 0$ and $y = a$, $x = 0$. *Ans.* $yz + ax - az = 0$.

14. Generated by a line parallel to the zx -plane intersecting the lines $x = 1$, $z = 0$ and $y = 2z$, $x = 0$. *Ans.* $xy + 2z - y = 0$.

15. Generated by a line parallel to the yz -plane intersecting the lines $y = 0$, $z = 0$ and $y = a$, $z = x$. *Ans.* $az = xy$.

16. Generated by a line parallel to the xy -plane intersecting the lines $x = a$, $y = 0$ and $z = x$, $y = b$. *Ans.* $yz - bx - ay + ab = 0$.

17. Generated by a line parallel to the zx -plane intersecting the lines $x = y = z$ and $x = a$, $z = 0$. *Ans.* $yz - xy = az - ay$.

18. Find the equation of the right circular conoid (see Ex. 23, p. 262) generated by a line parallel to the xy -plane intersecting the line $y = a$, $x = 0$ and the circle $x^2 + z^2 = a^2$, $y = 0$. Draw this surface.

189. Cylindrical coördinates. Given a point P in space, let us drop a perpendicular from P to a point M in the xy -plane. The position of P is evidently determined if we know the polar coördinates r, θ of M together with the z -coördinate MP . This combination of polar coördinates in the xy -plane with the cartesian z is the so-called *cylindrical* system. The coördinates are written in parenthesis in the order (r, θ, z) .

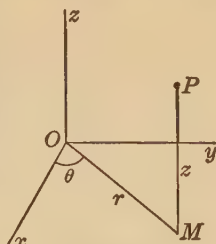


FIG. 141

Evidently the only formulas required for transformation from cylindrical to cartesian coördinates, or vice versa, are those given in § 107.

190. Spherical coördinates. The position of a point is determined if we know the length ρ of its radius vector, the angle ϕ from the z -axis to the radius vector, and the angle θ from the x -axis to the horizontal projection of the radius vector. The coördinates (ρ, θ, ϕ) are the *spherical coördinates* of P .

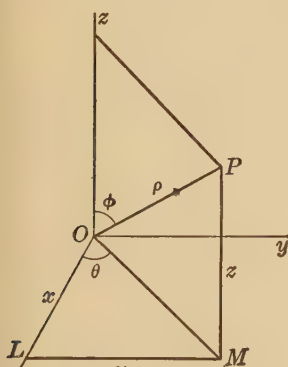


FIG. 142

A moment's thought shows that the system of coördinates used in geography is merely a slight modification of the system just described. The "longitude" of a point, measured from some standard meridian, is merely our angle θ ; the "latitude," in the northern hemisphere, is the complement

of ϕ . For this reason θ and ϕ are sometimes called *longitude* and *colatitude* respectively.

Since, in Fig. 142, $OM = \rho \sin \phi$, it appears by very simple trigonometry that

$$(1) \quad x = \rho \cos \theta \sin \phi, \quad y = \rho \sin \theta \sin \phi, \quad z = \rho \cos \phi.$$

Also

$$\rho^2 = x^2 + y^2 + z^2.$$

EXERCISES

Classify the following loci (cylindrical coördinates).

1. $r = 2$.
2. $\theta = \frac{1}{2}\pi$.
3. $r \sin \theta = a$.
4. $r \cos \theta = a$.
5. $r = 2z$.
6. $r^2 + z = 0$.
7. $z^2 - r^2 = a^2$.
8. $r^2 + 4z^2 = 4a^2$.
9. $2r^2 + z^2 = a^2$.
10. $r^2 - z^2 = a^2$.
11. $r = 2a \sin \theta$.
12. $r = 2a \cos \theta$.
13. Show that every equation in r and z (i.e. not involving θ) represents a surface of revolution around Oz .
14. Show that every equation in r and θ (not containing z) represents a vertical cylinder.
15. Show that every equation in z and θ (not containing r) represents a ruled surface (§ 186) with generators intersecting the z -axis at right angles.
16. Discuss the surface $z = a \tan \theta$. (Ex. 15.)
17. Discuss the surface $z \tan \theta = a$.
18. Show that if an equation in z and θ represents a (non-degenerate) quadric, the surface is a hyperbolic paraboloid. (Show that no other quadric has the property stated in Ex. 15.)

Classify the following surfaces (spherical coördinates), either directly or by transformation to cartesian coördinates.

19. $\phi = \frac{1}{6}\pi$.
20. $\rho = a$.
21. $\rho \cos \phi = a$.
22. $\rho \sin \phi = a$.
23. $\rho = 2a \cos \phi$.
24. $\rho \sin^2 \phi = a \cos \phi$.
25. $\rho^2 \cos 2\phi = a^2$.
26. $\rho^2 \cos 2\phi + a^2 = 0$.
27. Show in detail the derivation of equations (1), § 190.
28. Show that an equation in ρ and ϕ (not containing θ) represents a surface of revolution about Oz .
29. Show that an equation in ϕ and θ (not containing ρ) represents a cone with vertex at O .

CHAPTER XIX

CURVES

191. The locus of two simultaneous equations. As remarked in § 159, two surfaces intersect in general in a *curve*. The curve is represented analytically by the equations of the two surfaces simultaneously.

Of course the curve of intersection may be a plane curve, but ordinarily its points do not all lie in a plane, in which case it is called a *twisted curve*, or *skew curve*.

An infinite number of surfaces may evidently be passed through a given curve. Thus, while there is only one equation representing a given surface, *a curve may be represented by an infinite number of different pairs of equations*—viz. by the equations of any two of the surfaces having that curve as their intersection. (Cf. § 160.)

192. Surfaces through a given curve. Let the equations of any curve be denoted by

$$u = 0, \quad v = 0,$$

where u and v are certain expressions in x , y , and z . By reasoning analogous to that of § 161, we establish the

THEOREM: *The equation*

$$(1) \quad u + kv = 0$$

represents, for all values of the constant k , a surface passing through the curve

$$u = 0, \quad v = 0.$$

As noted in § 191, a curve may be represented analytically by the equations of any two of the infinite number of surfaces having that curve as their intersection. From

the given pair of equations it is often possible to derive a simpler pair from which the form and properties of the curve are more readily seen.

193. Projecting cylinders. If a perpendicular be dropped from a point P to a plane, the foot P' of the perpendicular is called the *projection* of P upon the plane, and the line PP' is the *projecting line*. If all the points of a

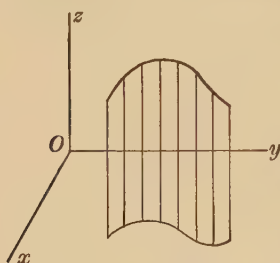


FIG. 143

curve in space be projected upon any plane, their projections form a certain curve in that plane: this curve is called the *projection* of the given curve, and the cylinder whose generators are the projecting lines is the *projecting cylinder* of the curve upon the plane. Figure 143

shows the projection of a curve upon the xy -plane, together with that portion of the projecting cylinder that lies between the curve and its projection.

Upon eliminating z between the equations of a curve we obtain a third equation which is satisfied whenever both of the original equations are satisfied, and therefore represents a surface through the given curve. Further, since the new equation does not contain z , it represents a cylinder perpendicular to the xy -plane—i.e. the xy -projecting cylinder of the given curve.* Hence:

To obtain the xy -projecting cylinder of a given curve, eliminate z between the equations of the curve; similarly for the other projecting cylinders.

An especially useful method of drawing a curve in space is to represent it as the intersection of two of its projecting cylinders.

* A formal proof of this process is easily written out by the method of example (a), § 188.

Examples: (a) Find the projecting cylinders of the curve

$$\begin{aligned}4x^2 + y^2 + 3z^2 &= 5, \\2x^2 + y^2 + z^2 &= 3,\end{aligned}$$

and draw the curve.

Eliminating x , y , and z in turn, we get the three projecting cylinders

$$(1) \quad y^2 - z^2 = 1,$$

$$(2) \quad x^2 + z^2 = 1,$$

$$(3) \quad x^2 + y^2 = 2.$$

Thus one projection of the curve is an arc of a hyperbola, while the other two are circular arcs. In Fig. 144 the curve is represented as the intersection of the circular

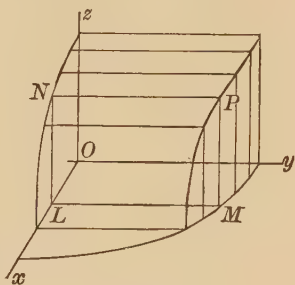


FIG. 144

cylinders (2) and (3); since the figure is symmetric with respect to all three coördinate planes it is sufficient to draw the part lying in the first octant. The construction is as follows: through any point L on Ox draw lines parallel to Oy and Oz intersecting the directing curves of the cylinders at M and N . Through M and N draw the generators of the cylinders respectively parallel to Oz and Oy ; their point of intersection P is a point of the curve.

(b) Find the projecting cylinders of the curve

$$x^2 + y^2 - z^2 = z, \quad x^2 + y^2 - z^2 = 1.$$

Subtracting the second equation from the first, we get

$$z = 1.$$

Substitution of this value of z in either of the original equations gives

$$x^2 + y^2 = 2.$$

Thus the curve is a right section of a circular cylinder: it is a circle of radius $\sqrt{2}$ with center in the z -axis, lying in the plane $z = 1$.

EXERCISES

1. Find the equation of a quadric surface passing through the curve $x^2 + y^2 + z^2 = 3$, $3x^2 + y^2 - z^2 = 3$ and through the point $(1, 1, 2)$. Sketch the three surfaces here occurring. *Ans.* $2x^2 + y^2 = 3$.

2. Find the equation of a quadric through the curve $y^2 = 4ax$, $x^2 = 2yz$ and the point (a, a, a) . *Ans.* $3x^2 - y^2 - 6yz + 4ax = 0$.

3. Find the equation of a quadric through the curve $x^2 - y^2 = z$, $x^2 + z^2 = y$ and the point $(1, 1, -1)$. Classify the three surfaces.

Ans. $y^2 + z^2 - y + z = 0$.

4. If $u = 0$ and $v = 0$ are two quadrics, prove that $u + kv = 0$ is also a quadric. Are there any exceptions?

5. If $u = 0$ and $v = 0$ are two spheres, prove that $u + kv = 0$ is also a sphere. Are there any exceptions?

6. If $u = 0$ is a quadric and $v = 0$ a plane, what is represented by $u + kv = 0$? By $u + kv^2 = 0$? Discuss exceptional cases.

7. Find the equation of a quadric surface through the curve $x^2 + y^2 - z^2 = 1$, $x + y + z = 3$ and the point $(3, 1, 1)$.

8. If $u = 0$, $v = 0$, $w = 0$, $t = 0$ are four planes, prove that the surface $uw + kwt = 0$ is a ruled quadric. (Show that the surface contains the family of lines $u = cw$, $cv + kt = 0$.)

9. If the four planes of Ex. 8 have a common point of intersection, prove that the surface $uw + kwt = 0$ is a cone. Where is its vertex?

10. If $u = 0$, $v = 0$, $w = 0$ are three planes, prove that the surface $uv + kw = 0$ is a ruled quadric. (Note the suggestion in Ex. 8.)

11. If the three planes of Ex. 10 have a common line of intersection, prove that the surface $uv + kw = 0$ is a quadric cylinder having the common line as one of its generators. (Take the common line as z -axis, and write out, by § 161, the equations of three planes through the z -axis.)

12. In Ex. 10, if the planes $u = 0$, $v = 0$ are parallel, prove that $uv + kw = 0$ is a cylinder having the parallel lines $u = 0$, $w = 0$ and $v = 0$, $w = 0$ as two of its generators.

Draw the following curves.

13. $y^2 = 4ax$, $z = y$.

14. $x^2 + z^2 = a^2$, $y = x$.

15. $y^2 + z^2 = a^2$, $x + y = a$.

16. $y^2 + 4z^2 = 2ay$, $y = x$.

17. $y^2 - x^2 = a^2$, $y = 2z$.

18. $yz = a^2$, $y = 3x$.

In each of the following cases, classify the given surfaces, find the projecting cylinders of the curve of intersection, and draw the curve.

19. $x^2 + 2y^2 + z^2 = 2a^2$, $3x^2 + 2y^2 - z^2 = 2a^2$.

20. $x^2 + y^2 + 3z^2 = 4$, $2y^2 + 9z^2 - x^2 = 8$.

21. $x^2 + 2y^2 + z^2 = 5$, $z^2 - 2x^2 - y^2 = 2$.

22. $3x^2 + 3y^2 + 2z^2 = 8$, $x^2 - 3y^2 + 6z^2 = 0$.

23. $x^2 - y^2 + 2z^2 = 0$, $2y^2 - x^2 - z^2 = 3$.

24. $2x^2 - y^2 - z^2 = 3$, $3y^2 - x^2 - 2z^2 = 1$.

25. $4x^2 - y^2 + z^2 = 0$, $4x^2 + 2y^2 - 5z^2 = 12$.

26. $x^2 - 2y^2 + z^2 = 0$, $x^2 + z^2 = 2x$.

27. $x^2 + y^2 + z^2 = 4$, $x^2 - y^2 - 3z^2 + 4 = 0$.

28. $x^2 + y^2 + 2z^2 = 5$, $x^2 + y^2 + z^2 = 1$. *Ans.* No intersection.

29. $x^2 + y^2 - 3z^2 = 9a^2$, $x^2 + 4y^2 + 9z^2 = 9a^2$. *Ans.* $(\pm 3a, 0, 0)$.

30. $x^2 + y^2 + z^2 = 4a^2$, $y^2 - x^2 - z^2 = 2a^2$.

31. $y^2 + z^2 = ax$, $2y^2 - z^2 = 2ax$.

32. $x^2 - 3y^2 - 3z^2 = 3$, $x^2 + 6y^2 + 6z^2 = 6$.

33. $3x^2 + y^2 - 2z^2 = 6$, $x^2 - y^2 + 10z^2 = 2$.

34. $2x^2 + y^2 - z^2 = 4$, $x^2 - y^2 = z^2$.

35. $x^2 + y^2 = z + 1$, $x^2 - y^2 = z - 1$.

36. $y^2 - 2y + x = 0$, $y^2 - 2y - z + 2 = 0$.

37. $x^2 + z^2 = 2y$, $y^2 + z^2 = 2y$. 38. $xy = 1$, $yz = 1$.

39. $x^2 + y^2 = z$, $x^2 + y^2 = z^2$. 40. $y^2 = zx$, $y^2 - x^2 = 2zx$.

41. $x^2 + y^2 = z$, $x^2 + y^2 = 2x$. 42. $z = xy$, $z^2 = x$.

Sketch the solid in the first octant bounded by the given surfaces.

43. $x^2 + z^2 = 1$, $y = x^2$.

44. $x = 1$, $x^2 = y + 2z$.

45. $x^2 + y^2 + z^2 = a^2$, $y = 2z$.

46. $x^2 + y^2 = a^2$, $z = y$, $z = 2y$.

47. $x = 1$, $z = x + y$, $y^2 = x$.

48. $z = xy$, $y = x^2$, $y = 1$.

49. $z = 1 - xy$, $y = 1 - x^2$.

50. $y^2 + x + z = 4$, $y + z = 1$.

51. $x^2 + az = a^2$, $x^2 + z^2 = ay$.

52. $z = x$, $y = x$, $x^2 = ay$.

53. Find the equation of the surface generated by revolving the curve $x^2 + y^2 + 2z^2 = 1$, $y = x$ about Oy . *Ans.* $2x^2 + 2z^2 = 1$.

54. Find the equation of the surface generated by revolving the curve $2x^2 - y^2 + z^2 = 1$, $y = x$ about Oz . *Ans.* $x^2 + y^2 + 2z^2 = 2$.

55. Find the equation of a cone with vertex at O and directing curve $x + y = 1$, $y^2 + z^2 = y$. *Ans.* $z^2 = xy$.

56. Prove that if the terms of the second degree in the equations of two quadrics are identical, the surfaces intersect in a plane curve.

194. Plane sections of a quadric surface. To investigate the intersection of a quadric surface and a plane, let us take the xy -plane parallel to the cutting plane, which of course involves no loss of generality. Thus we have to study the curve

$$(1) \quad Ax^2 + By^2 + Cz^2 + Dyx + Ezx + Fxy \\ + Gx + Hy + Iz + K = 0,$$

$$(2) \quad z = k.$$

Substituting $z = k$ in (1), we get

$$(3) \quad Ax^2 + By^2 + Fxy + (Ek + G)x \\ + (Dk + H)y + Ck^2 + Ik + K = 0,$$

which is a quadric cylinder through the given curve. Equations (2) and (3) taken together exhibit the curve as a right section of the cylinder (3); thus we have the

THEOREM: *Every plane section of a quadric surface is a conic.*

There is a limiting case, however, in which the section reduces to a single straight line; the student may investigate this case.

EXERCISES

1. Prove that every plane section of a sphere is a circle. (Take the cutting plane parallel to a coördinate plane. Is the proof still general?)

2. Prove that two spheres intersect in a circle.

3. Find the center and radius of the circle $x^2 + y^2 + z^2 = 1$, $x + y + z = 1$. *Ans.* $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}); \frac{1}{3}\sqrt{6}$.

4. Find the center and radius of the circle $x - y - z + 2 = 0$, $x^2 + y^2 + z^2 - 2x - 4z + 4 = 0$. *Ans.* $(\frac{2}{3}, \frac{1}{3}, \frac{7}{3}); \frac{1}{3}\sqrt{6}$.

5. Find the equation of the circular cylinder whose right section is the circle $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 + z^2 = 4x$. *Ans.* $y^2 + z^2 = 3$.

6. Find the equations of the tangent to the circle $x^2 + y^2 + z^2 = 9$, $x^2 + y^2 + z^2 - 3x - y - 2z = 0$ at the point $(1, 2, 2)$.

Ans. $3x + y + 2z = 9$, $x + 2y + 2z = 9$.

7. Solve Ex. 6 by another method.

8. Find the equations of the circle generated by revolving the point $(3, 1, 2)$ about the line $x = 1, y = 2$.

9. Find the equations of the circle generated by revolving the point $(0, 0, 0)$ about the line $x = y, z = y - 3$.

$$\text{Ans. } x + y + z = 0, x^2 + y^2 + z^2 = 2x + 2y - 4z.$$

10. A point is at the distance 4 from the point $(2, 3, 0)$ and at the distance 1 from the plane $2x + y + 2z = 6$. Find the equations of its locus, and discuss the curve.

11. Prove that the locus of a point whose distances from a fixed point and a fixed plane remain constant consists of two circles.

12. Find the equation of the cone whose vertex is at O and whose directing curve is the circle of Ex. 3.

$$\text{Ans. } xy + yz + zx = 0.$$

13. A *great circle* of a sphere is a circle cut from the sphere by a plane through the center. Find the equations of that great circle of the sphere $x^2 + y^2 + z^2 = 9$ that passes through $(3, 0, 0), (2, 1, -2)$.

$$\text{Ans. } x^2 + y^2 + z^2 = 9, 2y + z = 0.$$

14. Solve Ex. 13 by a second method.

15. Outline a method for finding the equations of a circle passing through three given points.

16. Find the equations of the circle through the points $(0, 2, -1), (4, 0, -3), (3, 2, -2)$.

$$\text{Ans. } x^2 + y^2 + z^2 - 2x + 6z + 1 = 0, x - y + 3z + 5 = 0.$$

17. Prove that every oblique plane section of a circular cylinder is an ellipse.

18. Prove that every plane section of an ellipsoid is an ellipse.

19. Prove that the plane $z = mx + b$ ($b \neq 0$) cuts from the circular cone $\frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{z^2}{c^2}$ an ellipse if $-\frac{c}{a} < m < \frac{c}{a}$; a parabola if $m = \pm \frac{c}{a}$;

a hyperbola if $m < -\frac{c}{a}$ or $m > \frac{c}{a}$.

20. A point moves so that its distances from a fixed line and a fixed plane remain constant. Prove that its locus is, in general, an ellipse. Discuss exceptional cases.

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